Hidden Dynamics for Piecewise Smooth Maps

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Abstract. We develop a hidden dynamics formulation of regularisation for piecewise smooth maps. This involves blowing up the discontinuity into an interval, but in contrast to piecewise smooth flows every preimage of the discontinuity needs to be blown up as well. This results in a construction similar to classic approaches to the Denjoy counterexample.

PACS numbers: 05.45.-a

Keywords: piecewise smooth dynamics, regularization, hidden dynamics, Sharkovskii's Theorem

Submitted to: *Nonlinearity*

1. Introduction

One approach to the study of piecewise smooth systems – systems for which the dynamics is defined by different smooth dynamical systems in different parts of the phase space – is to consider smooth approximations to these systems. For flows it is possible to 'blow up' the region between the different smooth systems in a way that respects the dynamics on either side of the discontinuity surface. This generalizes Filippov's ideas [4] and leads to 'hidden dynamics' in the blow up regions. The dynamics defined in this way is not unique, and this lack of uniqueness can be used to refine the modelling process [8]. Piecewise smooth maps, dynamical systems in discrete time, are equally important and theoretical results can be found in [1, 5, 6] with applications in [2, 10]. Current ideas for the regularization, or smoothing, of piecewise smooth maps involve filling in the gap in some way motivated by mechanical models [14] or by simple interpolation, or by allowing the map to be set-valued at the discontinuity [9].

In this paper we define a hidden dynamics approach to piecewise smooth one-dimensional maps. It turns out that in order to define a consistent blow up method which yields continuous maps, it is necessary not just to blow up the discontinuity surface, but all its preimages. This leads to a construction very similar to the classic approach to the Denjoy counterexample in circle maps [3], with hidden dynamics defined on these blown up intervals. If the extension of the maps to these intervals is as simple as possible we show that some properties of the hidden dynamics are independent of the details of the maps used to regularize the system. This shows that there is a level of robustness in the resulting continuous hidden dynamics. Moreover, in these examples all the new dynamics required to be compatible with a continuous model of the piecewise continuous map are restricted to the blown up intervals – these act as a sort of boundary layer containing the hidden dynamics in a well-controlled manner.

We begin with two key classes of piecewise smooth maps: degree one circle maps (section 2) and maps of the interval with a single point discontinuity (section 3). In both cases we define a class of continuous model equations and describe how the dynamics of simple examples (*minimal models*) have common properties, emphasising the systematic nature of the construction. These sections provide the main results of the paper. Two illustrative examples are given in sections 4 and 5.

2. Circle maps

Degree one maps of the circle can be defined by their lifts, $F : \mathbb{R} \to \mathbb{R}$ where $F(x+1) = F(x) + 1$ and the circle map is obtained by taking F modulo unity, or more accurately by defining $f(e^{2\pi i x}) = e^{2\pi i F(x)}$. We will consider the case when F is strictly increasing and F is continuous except at $c \in [0, 1)$ (and hence at $c + m$ for all $m \in \mathbb{Z}$) which is a point of discontinuity. With the notation

$$\lim_{y\uparrow x} F(y) = F(x_-), \quad \lim_{y\downarrow x} F(y) = F(x_+),$$

this implies that $F(c_-) < F(c_+)$. These were one of the first classes of piecewise smooth maps to be considered and important results are due to Keener [11] and Rhodes and Thompson [15, 16]. Our approach here is different, evoking more the techniques developed to describe the Denjoy counterexample [3, 12] and expanding Lorenz maps [7]. In this approach the circle is mapped to another circle but the image of one set of points is a set of intervals (the blow ups), but the dynamics away from these intervals is precisely the dynamics inherited from the original map. We are able to use this construction to create a hidden dynamics on these blown up intervals in such a way that the resulting map is continuous.

Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ denote the discontinuous map of the circle with lift F having the properties described above, and (with a slight abuse of notation) let $c \in \mathbb{T}^1$ denote the point of discontinuity. By a rotation of coordinates we may choose $c \in (0, 1)$ without loss of generality. Let

$$c_n = \{x \mid f^n(x) = c,\ n > 0,\ f^k(x) \neq c,\ 0 \leq k < n\}, \tag{1}$$

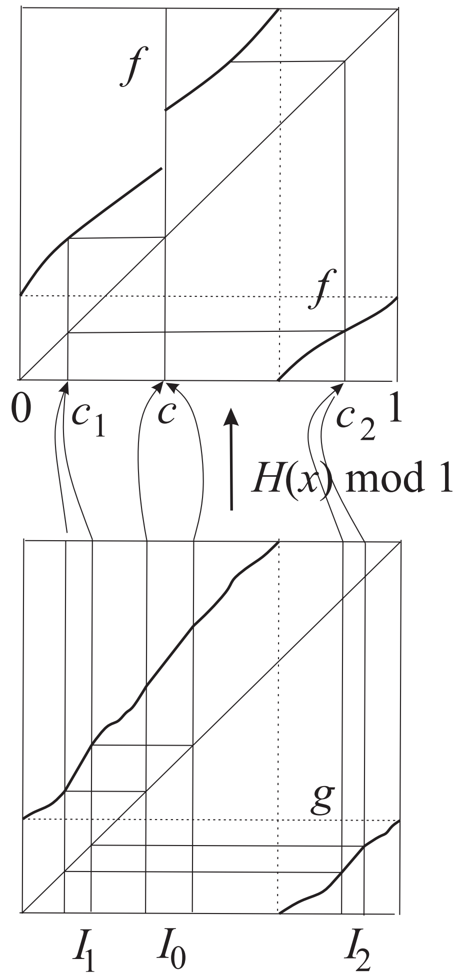

Figure 1. Schematic view of a circle map $f(x) = F(x) \bmod 1$ and an associated hidden dynamics model $g(x) = G(x) \bmod 1$.

and note that since F is strictly increasing c_n is either a singleton or empty, and if c_m is empty then c_k is empty for all $k > m$. Let N (possibly ∞) denote the number of non-trivial points c_n, with $c_0 = c$, and let ℓ_n, $n = 0, \ldots, N$, be a sequence of strictly positive numbers such that

$$\sum_{0}^{N} \ell_n = L < \infty. \tag{2}$$

We aim to 'open up' a small neighbourhood around c and each point c_n to define the lift G of a continuous circle map as shown schematically in Figure 1. This countable set of blown up intervals have lengths $(\ell_n/(1+L))$. By multiplying each of the ℓ_n by ε the total length εL goes to zero as $\varepsilon \to 0$ and the system 'tends' to the original piecewise smooth system.

Let

$$p(x) = \sum_{0 \le c_n < x} \ell_n, \quad P(x) = \sum_{0 \le c_n \le x} \ell_n, \tag{3}$$

and

$$\alpha(x) = \frac{1}{1+L}(x + p(x)), \tag{4}$$

with $p(0) = 0$, and extend p and P to functions of $\mathbb{R}$ using $p(x+m) = mL + p(x)$ for $m \in \mathbb{Z}$ and similarly for P, so $\alpha(x+1) = \alpha(x) + 1$. Define $I_n = [\alpha_n, \beta_n]$ where

$$\alpha_n = \alpha(c_n), \quad \beta(c_n) = \alpha(c_n) + \frac{\ell_n}{1+L} = \frac{1}{1+L}(c_n + P(c_n)).$$

Note that the intervals I_n occur in the same order as the points c_n in the interval $[0,1)$ and that they are disjoint. If x is not equal to c_n for some n we define our modified function $G(x)$ for $x \in [0,1)$ by

$$G(\alpha(x)) = \alpha(F(x)) = \frac{1}{1+L}(F(x) + p(F(x))). \tag{5}$$

Define G on I_n by

$$G(y) = h_n(y), \tag{6}$$

where $h_n : I_n \to \mathbb{R}$ is a continuous function satisfying the continuity conditions $h_n(\alpha_n) = \alpha_{n-1}$ and $h_n(\beta_n) = \beta_{n-1}$ where we have extended the definition at the index $n = 0$ using $\alpha_{-1} = F(c_-)$ and $\beta_{-1} = F(c_+)$.

This defines the map G on the interval $[0,1)$ and this can be extended to a map of the real line by defining $G(x+1) = G(x) + 1$.

Lemma 1 *Let F be the lift of a degree one circle map and suppose that F is strictly increasing and has a single discontinuity in $[0,1)$. Let G and N be as defined above. If $N < \infty$ then $G : \mathbb{R} \to \mathbb{R}$ is the lift of a continuous degree one circle map g and there is a continuous monotonic map $H : \mathbb{R} \to \mathbb{R}$ with $H(x+1) = H(x) + 1$ such that*

$$H(G(y)) = F(H(y)), \quad y \notin I_0. \tag{7}$$

If $N \leq \infty$ and h_k, $k = 0, \ldots, N$, are homeomorphisms then g is a continous, monotonic degree one circle map.

Proof: As noted earlier we may assume that 0 is not a preimage of c, so $G(0) = \alpha(F(0))$ and

$$\lim_{y \uparrow 1} G(y) = \alpha(F(1)) = \alpha(F(0) + 1) = \alpha(F(0)) + 1 = G(0) + 1,$$

i.e. G is continuous at 1 and hence at all integers. The definitions of G in (5) involves $G(y)$ with $y = \alpha(x)$. To invert this relation, and to extend it to $y \in I_n$, define H on $[0,1)$ by

$$H([\alpha(x), \beta(x)]) = x, \tag{8}$$

and extend H to a function of $\mathbb{R}$ using $H(x+1) = H(x) + 1$. It is not hard to see that H is continuous and monotonic (cf. [13]).

If $N < \infty$ then the number of intervals I_n is finite and by interpolating between the boundary values at α_n and β_n, G has the required properties by a similar but simpler argument to the case $N = \infty$ described below.

To prove G is continuous if $N = \infty$ and the h_k, $k = 0, 1, 2, \ldots$, are homeomorphisms, first note that if $y \in \text{int}(I_n)$ then G is continuous by the definition of h_n. If $y \notin \text{int}(I_n)$ then either $y = \alpha_n$, or $y = \beta_n$, or $y \notin I_n$. We will prove that G is left continuous at y in these cases and then that it is right continuous, hence that it is continuous.

If $y = \beta_n$ for some n then G is left continuous at y since h_n is continuous on I_n.

Now suppose that $y = \alpha_n$ for some n or $y \notin I_n$. Fix $\epsilon > 0$. We need a couple of preliminaries.

(i) By (2), for all $r > 0$ there exists $M(r) > 0$ such that

$$\frac{1}{1+L} \sum_{M(r)}^{\infty} \ell_n < r. \tag{9}$$

Moreover, for all $y \in (0, 1)$ with $x = H(y)$ there exist positive constants δ_1, δ_2 and δ_3 depending on y such that

(ii) if $0 < x - x_1 < \delta_1$ then $0 < F(x_-) - F(x_1) < \frac{1}{2}\epsilon(1 + L)$, from the monotonicity and continuity of the branches of F;

(iii) if $0 < x - x_1 < \delta_2$ and $c_n \in [x_1, x)$ then $n > M(\frac{1}{2}\epsilon) + 1$, since the c_n with $n < M(\frac{1}{2}\epsilon) + 1$ form a finite set; and

(iv) if $0 < y - y_1 < \delta_3$ then $0 < H(y) - H(y_1) < \min(\delta_1, \delta_2)$, from the montonicity and continuity of H.

So if $0 < y - y_1 < \delta_3$ then

$$\begin{aligned} 0 &\leq G(y) - G(y_1) \\ &\leq \tfrac{1}{1+L}\left(F(H(y)) - F(H(y_1)) + (p(F(H(y))) - p(F(H(y_1))))\right) \end{aligned}$$

and by (ii) and (iv), $0 < \frac{1}{1+L}(F(H(y)) - F(H(y_1))) < \frac{1}{2}\epsilon$, and by (i)-(iv) $0 \leq \frac{1}{1+L}(p(F(H(y))) - p(F(H(y_1)))) < \frac{1}{2}\epsilon$. Hence $0 \leq G(y) - G(y_1) < \epsilon$.

Hence G is left continuous at y. Since $G(y) - G(y_1) \geq 0$ g is increasing from below.

The proof that G is right continuous and increasing from above is entirely analogous.

Equation (7) and the properties of g now follow immediately from the definition of H and G.

□

Note that if $N = \infty$ and the h_k are not homeomorphisms then G need not be continuous. For example, choose h_k such that for all k, $h_k(u) = 1$ for some $u \in \text{int}(I_k)$. Consider $y = \alpha_n$ with $G(\alpha_n) \neq 1$ and such that c_n is an accumumation point of (c_k). Then there is an infinite sequence $u_k \to y$ such that $G(u_k) = 1$, but $G(y) \neq 1$.

The next result follows from the classic theorem of Poincaré for the existence of rotation numbers,

$$\rho(G) = \lim_{n\to\infty} \frac{1}{n}(G^n(x) - x)$$

for lifts of continuous, increasing degree one circle maps [3].

Theorem 2 *If F and G are defined as above and the (h_n) are homeomorphisms then the rotation number of the corresponding circle map g exists and is independent of (h_n).*

Proof: First note that if the maps (h_n) are homeomorphisms the $\alpha_n < \beta_n$ implies that G is strictly increasing and continuous and hence has a well-defined rotation number. By construction, $\cup_0^N I_n \neq [0,1)$ and so if there is a point $x \in [0,1)$ such that the corresponding circle map $g^n(x)$ never falls into one of these intervals the rotation number obtained using this point is independent of the choice of homeomorphisms (h_n) as required. If not, $g(I_0) \cup I_n \neq \emptyset$ for some n, in which case by construction $c_n \in [F(c_-), F(c_+)]$ (or one of its translates) and so $I_n \subseteq g(I_0)$ for all choices of the homeomorphism h_0. Hence $I_0 \subseteq g^{n+1}(I_0)$ for every choice of the homeomorphisms (h_n) and the order of the points is the same, so every map g has the same (rational) rotation number.

□

Rhodes and Thompson [16] prove results about the continuity of the rotation number for continuous (appropriately defined) families of discontinuous maps of the circle which could also be approached using our techniques, but we will not develop this extension here.

3. Maps of the interval with a single discontinuity

The same idea can be used for piecewise smooth maps of the interval with a single discontinuity and which are monotonic on each continuous branch. Here there is an added complication because the continuous branches may be either increasing or decreasing, whereas in the circle map case both are increasing.

Definition 1 *A monotonic single discontinuity map (MSDM) is a map $f : [0,1] \to [0,1]$ with discontinuity $c \in (0,1)$ such that*

$$f(x) = \begin{cases} f_0(x) & \text{if } 0 \leq x < c \\ f_1(x) & \text{if } c < x \leq 1, \end{cases}$$

where f_0 and f_1 are continuous monotonic functions and $f_0(c) \neq f_1(c)$.

Let

$$C_n = \{x \mid f^n(x) = c,\ f^k(x) \neq c,\ 0 \leq k < n\}.$$

Note that if C_n is non-empty then $C_n = \{c_{n,1}, \dots, c_{n,r_n}\}$ with $r_n \leq 2^n$. Define N such that C_n is non-empty if $n \leq N \leq \infty$. To include the discontinuity itself let $c = c_{0,1}$. Define as before lengths $\ell_{n,r}$ chosen so that

$$\sum_{n=0}^{N} \sum_{1}^{r_n} \ell_{n,r} = L < \infty.$$

Given x, let

$$d(x) = \{(n,r) \mid c_{n,r} < x\}, \quad D(x) = \{(n,r) \mid c_{n,r} \leq x\},$$

with

$$(1+L)a(x) = x + \sum_{(n,r)\in d(x)} \ell_{n,r}, \quad (1+L)b(x) = x + \sum_{(n,r)\in D(x)} \ell_{n,r}.$$

If $n \geq 0$ let $I_{n,r} = [a_{n,r}, b_{n,r}]$ where $a_{n,r} = a(c_{n,r})$ and

$$b_{n,r} = b(c_{n,r}) = a_{n,r} + \ell_{n,r}.$$

By definition $f(c_{n,r}) = c_{n-1,r'}$ if $n > 0$, some $r' \in \{1, \ldots, r_{n-1}\}$. Now define maps $h_{n,r} : I_{n,r} \to \mathbb{R}$ such that end-points map to the end points of $I_{n-1,r'}$ in a way which ensures continuity: $h_{n,r}(a_{n,r}) = a_{n-1,r'}$ and $h_{n,r}(b_{n,r}) = b_{n-1,r'}$ if either $c_{n,r} < c$ and f_0 is increasing or $c_{n,r} > c$ and f_1 is increasing, and $h_{n,r}(a_{n,r}) = b_{n-1,r'}$ and $h_{n,r}(b_{n,r}) = a_{n-1,r'}$ if either $c_{n,r} < c$ and f_0 is decreasing or $c_{n,r} > c$ and f_1 is decreasing.

Finally, we need to define the map for $x \in I_{0,1}$. This is more complicated than in the circle map case because the conditions for continuity and spanning the appropriate union of $I_{n,r}$ do not coincide. Let

$$U_1 = \begin{cases} a(f_0(c)) & \text{if } \; f_0(c) < f_1(c) \\ a(f_1(c)) & \text{if } \; f_0(c) > f_1(c) \end{cases}, \tag{10}$$

and

$$U_2 = \begin{cases} b(f_0(c)) & \text{if } \; f_0(c) > f_1(c) \\ b(f_1(c)) & \text{if } \; f_0(c) < f_1(c) \end{cases}. \tag{11}$$

and let $h_{0,1} : I_{0,1} \to \mathbb{R}$ be a map which sends the end points of $I_{0,1}$ to $a(f_0(c))$ and $b(f_1(c))$ preserving continuity, i.e. $h_{0,1}(a(c_{0,1})) = a(f_0(c))$ and $h_{0,1}(b(c_{0,1})) = b(f_1(c))$, and which also maps two points $u_1, u_2 \in I_{0,1}$ to U_1 and U_2 so that $h_{0,1}(u_i) = U_i$, $i = 1, 2$.

We are now in a position to define the hidden dynamics maps associated with f.

Theorem 3 *Let f be a MSDM and define $g : [0,1] \to [0,1]$ by*

$$g(y) = \begin{cases} a(f(x)) & \text{if } \; y = a(x), \; x \neq c_{n,r}, \\ h_{n,r}(y) & \text{if } \; y \in I_{n,r}. \end{cases} \tag{12}$$

If the maps $(h_{n,r})$ are homeomorphisms for $n \geq 1$ then g is a continuous map of the interval and there exists a monotonic surjection $h : [0,1] \to [0,1]$ such that $h(g(y)) = f(h(y))$ if $y \notin I_{0,1}$.

The proof follows the proof of Lemma 1 closely and is omitted. The condition on the (u_i) is to ensure that if an end-point $a(c)$ or $b(c)$ maps to an end-point of one of the sets $I_{n,r}$ under $h_{n,r}$, then the whole of that set $I_{n,r}$ is contained in the image of $I_{0,1}$ under $h_{0,1}$. This is automatic if left end-points are mapped to left end-points and vice versa (i.e. if $A = a(f_0(c))$ and $B = b(f_1(c))$ as in the circle map case), but is an extra condition in general. The examples of sections 4 and 5 provide further explanation for this complication.

Definition 2 *If the maps $(h_{n,r})$ are homeomorphisms for $n \geq 1$ and the map $h_{0,1} : I_{0,1} \to [A, B]$ is a surjection with the fewest possible number of critical points consistent with its definition then we will say that the resulting map g is a minimal model of f.*

A kneading invariant can be associated with a map consisting of a finite number of monotonic continuous branches. For continuous maps the kneading invariant is effectively the symbolic description of the dynamics of each critical (turning) point, i.e. it describes the sequence of monotonic branches that the orbit passes through, see [12] for details. Two continuous maps of the interval with the same kneading invariants have effectively the dynamics up to issues such as the stability of periodic orbits, the existence of homtervals and the existence of some period-doubled orbits [12], p. 103. Note that the maps g defined above are bimodal (two turning points) if f_0 and f_1 have the same orientation, whilst they are unimodal if the two branches have different orientations.

Theorem 4 *Let f be a MSDM and suppose that g_1 and g_2 are two minimal models of f. Then g_1 and g_2 have the same kneading invariants.*

Proof:

Case (i): f_0 is increasing and f_1 is decreasing (the case f_0 decreasing and f_1 increasing is equivalent after reversing the direction of x, i.e. $x \to 1 - x$).

If $f_0(c) > f_1(c)$ then from (10) and (11), $U_1 = a(f_1(c))$ and $U_2 = b(f_0(c))$. If $a(f_0(c)) = b(f_0(c))$, i.e. if $f_0(c)$ is not a preimage of c, then $I_{0,1} = [a(c), b(c)] = [u_2, u_1]$ and the minimal $h_{0,1}$ is a continuous strictly decreasing surjection and g is a unimodal map with critical point $a(c)$. Since $f_0(c)$ is not a preimage of c, $g^n(a(c))$ does not fall in any of the blow up intervals and the combinatorial position of the iterates is the same regardless of the choice of minimal $h_{0,1}$.

If $f_0(c)$ is a preimage of c then $g(a(c)) \neq U_2$ and we choose $u_2 \in (a(c), b(c))$ and define $g(u_2) = U_2$ with $h_{0,1}$ increasing on $(a(c), u_2)$ and decreasing on $(u_2, b(c))$ so g has a critical point at u_2 and $g(a(c)) = a(f_0(c))$. Now, $g(u_2)$ is the right hand end point of $I_{n,r}$ for some (n, r) and so $g^{n+1}(u_2)$ is an end-point of $I_{0,1}$. By assumption $g(a(c))$ is the left end point of $I_{n,r}$ and so if $g^{n+1}(u_2) = a(c)$ then the orbit of u_2 is eventually periodic with an order determined by the orbit of the relevant $I_{n,r}$, which is independent of the choice of minimal map.

If $g^{n+1}(u_2) = b(c)$ then either $b(c)$ is also an end point of a set $I_{m,s}$ and we can argue similarly, or it is not, in which case the order is determined by the iterates of $f_1(c)$ which is not a preimage of c. In either case the order of the points in the orbit of u_2 are determined independent of the choice of minimal map.

If $f_0(c) < f_1(c)$ then $U_1 = a(f_0(c))$ and $U_2 = b(f_1(c))$. The argument is now as before: the case in which $f_1(c)$ is not a preimage of c is easy as $a(f_1(c)) = b(f_1(c))$, whilst otherwise we can choose $u_2 \in (a(c), b(c))$ and define $g(u_2) = U_2$ with $h_{0,1}$ increasing on $(a(c), u_2)$ and decreasing on $(u_2, b(c))$ and $g(b(c)) = a(g_1(c))$. Then g is a unimodal map with critical point at u_2. The argument is now similar to the argument above.

Case (ii): If f_0 and f_1 are increasing, then for non-trivial dynamics $f_0(c) > f_1(c)$ and $h_{0,1}$ is decreasing on $I_{0,1}$. The maps g hence have two critical points, the smaller, $a(c)$, being a maximum and the larger, $b(c)$, a minimum. There is a pair of critical orbits, but the arguments of the first case still hold to show that the kneading invariants are independent of the choice of homeomorphisms $(h_{n,r})$.

Case (iv): If f_0 and f_1 are decreasing, then for non-trivial dynamics $f_0(c) < f_1(c)$ and $h_{0,1}$ is increasing on $I_{0,1}$. The maps g hence have two critical points, the smaller being a minimum and the larger a maximum. As in the previous case there is a pair of critical orbits, but the arguments of the first case still hold to show that the kneading invariants are independent of the choice of the maps $(h_{n,r})$.

□

Recall that the Sharkovskii order of the positive integers is

$$\begin{array}{l} 1 \prec 2 \prec 2^2 \prec \ldots 2^n \prec \ldots \\ \ldots 2^{k+1}.9 \prec 2^{k+1}.7 \prec 2^{k+1}.5 \prec 2^{k+1}.3 \prec \ldots \\ \ldots 2^k.9 \prec 2^k.7 \prec 2^k.5 \prec 2^k.3 \prec \ldots \\ \ldots 9 \prec 7 \prec 5 \prec 3. \end{array}$$

Sharkovskii's Theorem states that if f is a continuous map and f has an orbit of period q then it has an orbit of period p for all $p \prec q$ in the Sharkovskii order [12].

Corollary 5 *If f is a MSCM with minimal model g. Then either the set of periods of g is a finite set $\{1, 2, 4, \ldots 2^m\}$ and for every minimal model of f, m is either k or $k+1$ for some $k \geq 0$, or every minimal model g has the same set of periods.*

Proof: Since g is continuous Sharkovskii's Theorem holds for g. The set of periods that exist for the maps g are determined by their kneading invariants. Note that in the one case of ambiguity for the kneading invariant, where the (smooth) map may have an attractor of period p or $2p$ for some p. Then if the attractor has period $2p$ then it is an orbit obtained by period-doubling of an orbit of period p which also exists for the map. If p is not a power of two then $2p$ is to the left of p in the Sharkovskii order and a (non period-doubled) orbit of period $2p$ exists, otherwise the kneading invariant defines the set of periods up to the ambiguity noted before.

□

4. Example: Lorenz maps

Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } \ 0 \leq x < \frac{1}{2} \\ x - \frac{1}{2} & \text{if } \ \frac{1}{2} < x \leq 1, \end{cases} \tag{13}$$

with the value of $f(x)$ at $x = \frac{1}{2}$ left ambiguous. This can be seen as a continuous circle map with rotation number $\frac{1}{2}$ by taking $f(\frac{1}{2}) = 0$ and restricting to $[0,1)$, but as a Lorenz map it has trivial dynamics: all points apart from $C = \{0, \frac{1}{2}, 1\}$ are periodic with period two, and the fate of this set depends on how the value of the map at the discontinuity is assigned.

To construct a minimal model, note that the discontinuity is at $c = \frac{1}{2}$; this has two preimages: $c_{1,1} = 0$ and $c_{1,2} = 1$, and these points have no further preimages apart from possibility of the discontinuity itself, which is not included by definition.

We start by blowing up these intervals. To emphasise that there is no need for symmetry in this process we will choose

$$\ell_{0,1} = \tfrac{1}{2}, \quad \ell_{1,1} = \tfrac{1}{6}, \ell_{1,2} = \tfrac{1}{3},$$

so $L = 1$, the scaling factor $1 + L = 2$ and so we expect $I_{0,1}$ to have length $\frac{1}{4}$, $I_{1,1}$ to have length $\frac{1}{12}$ and $I_{1,2}$ to have length $\frac{1}{6}$. Moreover, $f_0(c) = 1 > f_1(c) = 0$ so

$$a(f_0(c)) = \tfrac{1}{2}(1 + \tfrac{1}{6} + \tfrac{1}{2}) = \tfrac{5}{3}, \quad b(f_0(c)) = 1,$$

$$a(f_1(c)) = 0, \quad b(f_1(c)) = \tfrac{1}{2}(0 + \tfrac{1}{6}) = \tfrac{1}{12}.$$

Since $f_0(c) > f_1(c)$, $A = 0$ and $B = 1$.

Now from (12), if $0 < x < \frac{1}{2}$ then $y = a(x) = \frac{1}{2}(x + \frac{1}{6})$ (as only $c_{1,1}$ is to the left of x) and $g(y) = \frac{1}{2}(x + \frac{1}{2}) + (\frac{1}{2} + \frac{1}{6})$ as $x + \frac{1}{2} > \frac{1}{2}$. Hence $x = 2y - \frac{1}{6}$ and $x > 0$ implies that $y > \frac{1}{12}$, $x < \frac{1}{2}$ implies that $y < \frac{1}{3}$ and so

$$g(y) = \tfrac{1}{2}(2y - \tfrac{1}{6} + \tfrac{7}{6}) = y + \tfrac{1}{2}, \quad \text{if} \quad \tfrac{1}{12} < y < \tfrac{1}{3}.$$

Similarly, if $\frac{1}{2} < x < 1$ then $y = a(x) = \frac{1}{2}(x + \frac{2}{3})$ and $g(y) = \frac{1}{2}(x - \frac{1}{2}) + \frac{1}{6}$ so

$$g(y) = y - \tfrac{1}{2}, \quad \text{if} \quad \tfrac{7}{12} < y < \tfrac{5}{6}.$$

Now

$$I_{1,1} = [0, \tfrac{1}{12}], \quad I_{0,1} = [\tfrac{1}{3}, \tfrac{7}{12}], \quad I_{1,2} = [\tfrac{5}{6}, 1],$$

and so $h_{1,k}$ $k = 1, 2$ may be chosen to be an affine map from $I_{1,k}$ to $I_{0,1}$. The final map used to define g is $h_{0,1} : [\frac{1}{3}, \frac{7}{12}] \to [0, 1]$ which satisfies the continuity conditions $h_{0,1}(\frac{1}{3}) = \frac{5}{6}$ and $h_{0,1}(\frac{7}{12}) = \frac{1}{12}$ and which, to be a minimal model must have a minimum value of 0 at u_1 and a maximum of 1 at u_2; for the smallest possible number of turning points $u_2 < u_1$ and so we will choose

$$u_2 = \tfrac{5}{12}, \quad u_1 = \tfrac{1}{2}.$$

Putting these together with a piecewise linear map we obtain $g : [0, 1] \to [0, 1]$ defined by

$$g(y) = \begin{cases} 3y + \frac{1}{3} & \text{if} \ \ 0 \leq x \leq \frac{1}{12} \\ y + \frac{1}{2} & \text{if} \ \ \frac{1}{12} \leq y \leq \frac{1}{3} \\ 2y + \frac{1}{6} & \text{if} \ \ \frac{1}{3} \leq y \leq \frac{5}{12} \\ -12y + 6 & \text{if} \ \ \frac{5}{12} \leq y \leq \frac{1}{2} \\ y - \frac{1}{2} & \text{if} \ \ \frac{1}{2} \leq y \leq \frac{7}{12} \\ y - \frac{1}{2} & \text{if} \ \ \frac{7}{12} \leq y \leq \frac{5}{6} \\ \frac{3}{2}y - \frac{11}{12} & \text{if} \ \ \frac{5}{6} \leq y \leq 1 \ . \end{cases} \tag{14}$$

The two occurrences of $y - \frac{1}{2}$ are included to emphasise that in one case the definition is in $I_{0,1}$ and the other is in the gap between $I_{0,1}$ and $I_{1,2}$.

As expected from the proof of Theorem 4 the dynamics (independent of the particular functions chosen for the minimal map) is defined by a Markov partition of the seven intervals on which the map has been defined. If we label these $I_1 = [0, \frac{1}{12}]$, $J_1 = [\frac{1}{12}, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{5}{12}]$, $I_3 = [\frac{5}{12}, \frac{1}{2}]$, $I_4 = [\frac{1}{2}, \frac{7}{12}]$, $J_2 = [\frac{7}{12}, \frac{5}{6}]$, $I_5 = [\frac{5}{6}, 1]$, so $I_{0,1} = I_2 \cup I_3 \cup I_4$, then we can read off from (14)

$$\begin{array}{ll} g(J_1) = J_2, & g(J_2) = J_1, \\ g(I_1) = I_2 \cup I_3 \cup I_4, & g(I_5) = I_1, \\ g(I_2) = I_5, & g(I_4) = I_1, \\ & g(I_3) = J_1 \cup J_2 \cup I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5. \end{array} \tag{15}$$

From this, using standard techniques, we can read off the periodic orbits of the map. The only periodic orbits in $J_1 \cup J_2$ have period two (due to the degeneracy of the equations all points here have period two), and these correspond to the orbits of the original Lorenz map which do not involve the discontinuity. All other recurrent orbits are in the complement of these, and we can read off immediately that there are two period three orbits ($I_3^2 I_5$ and $I_3^2 I_1$) and so there are orbits of all periods. The much richer information in (15) will be true of all minimal models of f. Note how all this dynamics lies in the boundary layer opened up by the regularization construction (the intervals labelled I_k).

Finally, we can see that the orbits of the critical points $u_2 < u_1$ are

$$u_2 = \tfrac{1}{2} \to 1 \to \tfrac{1}{3} \to \tfrac{5}{6} \to \tfrac{1}{3} \to \dots$$

and

$$u_1 \to \tfrac{7}{12} \to 0 \to \tfrac{1}{3} \to \tfrac{5}{6} \to \tfrac{1}{3} \to \dots$$

and all minimal models will have the same asymptotically periodic structure.

Minimal models of more general Lorenz maps will have the same bimodal structure.

5. Example: unimodal maps

Again we will use an example for which the dynamics of the piecewise smooth map is simple; if a map has positive entropy then the number of preimages of the discontinuity is infinite which makes it harder to do more than describe the dynamics implicitly. In this case however we will not specify the maps precisely, but only indicate the constructions.

Consider the map $f : [0, 1] \to [0, 1]$ as shown in Figure 2a. This has a discontinuity at c and is formed of a monotonic increasing branch f_0 in $x < c$ and a monotonic decreasing branch f_1 in $x > c$. There is a point $d \in (0, c)$ such that

$$f(0) = d, \quad f(d) = c, \quad f_0(c) = 1, \quad f_1(c) = d, \quad \text{and} \quad f(1) = 0. \tag{16}$$

Let $K_1 = (0, d)$, $K_2 = (d, c)$ and $K_3 = (c, 1)$, then $f(K_1) = K_2$, $f(K_2) = K_3$ and $f(K_3) = K_1$ so either there is a period three orbit in the K_k or the dynamics limits on a period 3 orbit on the continuous extension of the maps. Note that in some sense $F_1(c)$ has 'period' three: $f_1(c) \to d \to c \to f_1(c)$, whilst $f_0(c)$ has 'period' four: $f_0(c) = 1 \to 0 \to d \to c \to f_0(c)$.

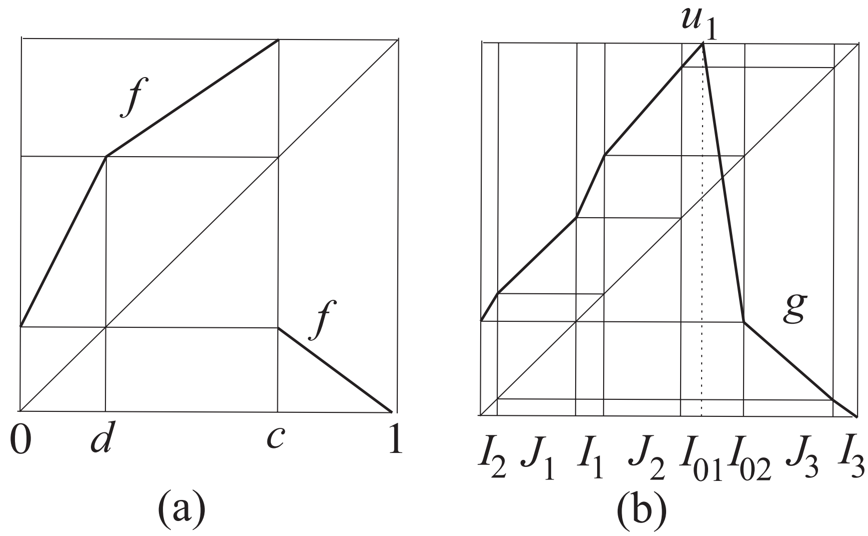

Figure 2. (a) The piecewise smooth map f; (b) a minimal model of f.

The discontinuity has only three preimages: 0, d and 1, so we begin the construction of the model map by specifying lengths ℓ_0, ℓ_1, ℓ_2 and ℓ_3 with total length L. Then define

$$
\begin{aligned}
&I_2 = [0, \ell_2/(1+L)], \quad I_1 = [(d+\ell_2)/(1+L), (d+\ell_1+\ell_2)/(1+L)],\\
&I_0 = [(c+\ell_1+\ell_2)/(1+L), (c+\ell_0+\ell_1+\ell_2)/(1+L)],\\
&I_3 = [(1+\ell_0+\ell_1+\ell_2)/(1+L), 1].
\end{aligned}
$$

The model map has $g(I_3) = I_2$, $g(I_2) = I_1$ and $g(I_1) = I_0$, with g a continuous strictly increasing surjection on each interval.

On I_0, $f_0(c) > f_1(c)$ implies that $U_1 = a(f_1(c))$ and $U_2 = b(f_0(c))$ with the notation of section 3. Hence $U_1 = a(d)$ and we need to define $u_2 \in (a(c), b(c))$ such that $g(u_2) = b(f_0(c)) = 1$. Then take $g : I_0 \to [a(c), 1]$ to be unimodal with $g(a(c)) = a(f_1(c)) = (d+\ell_2)/(1+L)$, $g(u_1) = 1$ and $g(b(c)) = a(f_0(c)) = (1+\ell_0+\ell_1+\ell_2)/(1+L)$.

On the intervals $J_1 = [\ell_2/(1+L), (d+\ell_2)/(1+L)]$, $J_2 = [(d+\ell_1+\ell_2/(1+L), (c+\ell_1+\ell_2)/(1+L)]$ and $J_3 = [(c+\ell_0+\ell_1+\ell_2)/(1+L), (1+\ell_0+\ell_1+\ell_2)/(1+L)]$, g is defined via (12) and the maps restricted to J_1 is a continuous monotonic surjection onto J_2, and similarly from J_2 to J_3 and J_3 to J_1.

The resulting Markov graph is shown in Figure 3 reflecting the dynamics obtained by dividing I_0 into two intervals: $I_{0,1} = [a(c), u_2]$ and $I_{0,2} = [u_2, b(c)]$:

$$
\begin{aligned}
&g(J_1) = J_2, \quad g(J_2) = J_3, \quad g(J_3) = J_1,\\
&g(I_2) = I_1, \quad g(I_1) = I_0, \quad g(I_3) = I_2,\\
&\qquad\qquad g(I_{0,1}) = I_3, \quad g(I_{0,2}) = I_{0,1} \cup I_{0,2} \cup I_1 \cup J_2 \cup J_3 \cup I_3.
\end{aligned} \tag{17}
$$

This shows immediately that there is a periodic orbit of period three in $J_1 \cup J_2 \cup J_3$ and no other recurrent dynamics – this is the ‘period three’ we noted for f. However, since g is continuous Sharkovskii’s theorem implies the existence of periodic orbits of every period, and these exist in $I_0 \cup I_1$. Hence the additional orbits exist in the boundary layer created by the blow up process.

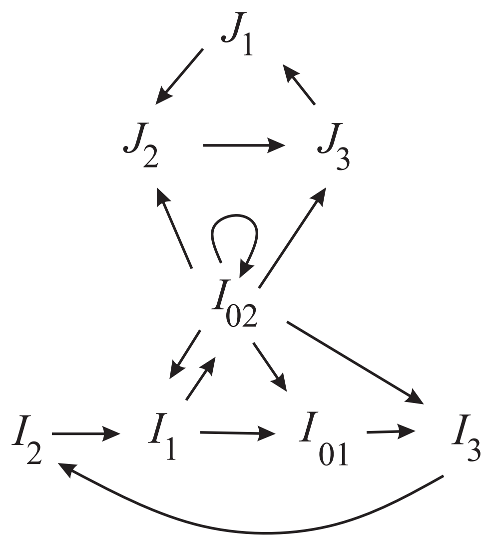

Figure 3. Markov graph associated with (17).

The critical point of g is u_2 and this has orbit

$$\begin{aligned} &u_2 \to 1 \to 0 \to (d+\ell_2)/(1+L) \to a(c) \\ &\to (1+\ell_0+\ell_1+\ell_2)/(1+L) \to \ell_2/(1+L) \\ &\to (d+\ell_1+\ell_2)/(1+L) \to b(c) \to (d+\ell_2)/(1+L) \to a(c)\ldots \end{aligned}$$

i.e. the critical point itself has asymptotic period *six*, although the kneading invariant (coding by C for u_1, R for iterates above u_1 and L for iterates below u_1) is $CRL(LRL)^\infty$, and this will be replicated in any minimal model of f.

6. Conclusion

The technique described here could be extended to piecewise continuous maps with a countable set of discontinuities, since the preimages are a countable set of countable sets, hence countable and the lengths of the blow up intervals can still be chosen to have a finite sum.

The process of opening up intervals of discontinuity as described above requires a countable set of alterations to the maps (but can be done on length ϵ which can then tend to zero). The effect is to define a hidden dynamics which, if the functions are as simple as consistent with continuity, restrict the extra dynamics needed to create continuous models in a boundary layer structure. If the maps on the blown up preimages are homeomorphisms then our construction gives a continuous model, but *any* continuous map on the blow up of the point of discontinuity which respects the boundary conditions can be used. This seems to correspond to the freedom noted in the hidden dynamics of piecewise smooth flows [8], and we have introduced minimal models to describe the simplest hidden dynamics.

There are many directions in which these ideas could be taken. Clearly one would like to develop a theory of how the hidden dynamics varies for parametrized families of piecewise smooth maps. From a theoretical point of view the differentiability of the minimal models is also interesting, and this may become important when considering families of maps. This is straightforward if the number of blown up intervals is finite as in the examples of sections 4 and 5. Extensions to higher dimensional maps are complicated by the possible geometry of the intersections of the discontinuity surfaces and their preimages. A consistent formulation in this case would be interesting. Ultimately, this only becomes useful if it can be applied to examples, and the examples presented here have been chosen to bring out the theory rather than the potential applications.

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