

PAUSING FOR ARBITRARILY LONG TIMES IN DYNAMICAL SYSTEMS

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ABSTRACT. It is well known that continuity in dynamical systems is not sufficient to guarantee uniqueness of solutions, but less obvious is that non-uniqueness can carry internal structure useful to characterize a system's dynamics. The non-uniqueness that concerns us here arises when an isolated non-differentiability of a flow results in spatial or temporal ambiguity of solutions. Spatial ambiguity can render a flow set-valued after a specific event, and non-trivial examples are increasingly being seen in models of switching occurring in electronic or biological systems. Temporal ambiguity can mean that the same spatial trajectory may be traversed in different times, making an arbitrarily long pause at the non-differentiable point. We focus here on temporal indeterminacy and the extent to which it can be resolved. To investigate the typical forms we take representative examples of the different conditions (non-differentiability, discontinuity, or singularity) under which it occurs.

1. INTRODUCTION

A dynamical system usually consists of a set of ordinary differential equations describing how a system changes with time, together with some initial condition, and differentiability of those equations is a key requirement for existence and uniqueness of solutions. If the equations are non-differentiable at a particular point then the system may still be solvable, but the solutions may be non-unique. When this happens, those solutions do not become meaningless, but instead contain information about the set of possible behaviours a system can exhibit. In some cases uniqueness can be partially restored by re-scaling of system variables. We show here how such re-scaling reveals, in a wide set of circumstances, arbitrarily long pauses in motion, sometimes with subsequent choice between alternative trajectories.

A system can be continuous and differentiable almost everywhere, with unique solutions, but with loss of differentiability at isolated points

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or thresholds that lead to isolated loss of uniqueness. This can result in spatial indeterminacy, where many solutions pass through the same point, or temporal indeterminacy, where many solutions lie along the same trajectory but with different travel times, manifesting as arbitrarily long pauses, through certain states.

The indeterminacy of solutions of dynamic systems at ill-behaved points is of course well known, but the commonly cited cases, such as the equation $\dot{x} = \sqrt{|x|}$ at $x = 0$, are not typically studied for their own sake, but only given as examples of the breakdown of determinacy that follows from violating so-called Lipschitz continuity (see e.g. [8, 6, 18, 19, 24, 27] or any standard textbook on dynamics or ordinary differential equations). Examples of systems that are only piecewise-differentiable are becoming increasingly common in applications, though often in complicated situations that disguise their common features. One can find isolated examples in classical mechanics of non-smooth equations of motion giving non-unique dynamics, such as a particle following a non-Lipschitz continuous path in [3]. Non-trivial instances of uniqueness breaking down due to a pointwise non-differentiability of a flow have arisen in genetic models [21, 11], the classical mechanical Painlevé paradox [23, 5], and studies of sliding explosions and new forms of chaos [12].

A number of physical situations are characterized by pauses between intervals of activity, such as earthquakes, or the spiking of neurons in brain activity, both characterized by dynamic episodes interspersed by static phases of varying length. Although such stationary periods seem dynamically simple, they appear to be hard to model, at least to obtain reliable predictions of their duration, suggesting an apparently random or indeterminate element to the times between dynamic events. This makes it interesting to revisit more generally the question of how such indeterminacy arises in dynamical systems.

To help bring some generality and clarity to the issue, we present a number of local models here that help reveal the ways that indeterminacy arises, focusing on cases that manifest as an arbitrarily long pause in motion, and we present general methods that can be used to analyze them. We will show certain classes of system where uniqueness of solutions can be restored almost everywhere, except at certain parameter values and along certain trajectories, where solutions can pause for in particular states for arbitrarily long times. Thus pausing becomes an indicator of structural changes occurring in the underlying system.

We begin by describing a general approach for resolving the non-uniqueness by parameter re-scalings and asymptotic balance. This is

then applied to a number of systems exemplifying different kinds of singularities, showing the extent to which their associated non-uniqueness can be resolved.

Our general approach is set out in section 2, and applied to revisit the classic example in section 3. Examples of a discontinuous system with one or two switches are given in section 4, and a classic applied example, the Painlevé paradox for a rod undergoing frictional impact, is considered in a new light in section 5. Finally we study examples of pausing at a singularity of a vector field in section 6, including an example from genetic regulatory models. Concluding remarks are made in section 7.

2. PAUSING SOLUTIONS AND BLOW-UP

The approach used to resolve uniqueness in this paper can be laid out schematically in one dimension. The examples in the following sections will extend this to higher dimensions. Let $x \in \mathbb{R}$ and consider a dynamical system

$$\dot{x} = f(x) = \begin{cases} f^-(x) & \text{if } x < 0, \\ f^+(x) & \text{if } x > 0, \end{cases} \quad (1)$$

where $f^\pm(x) > 0$ for $x \neq 0$. The functions f^+ and f^- are continuous on the disjoint domains $x \geq 0$ and $x \leq 0$, and differentiable on $x > 0$ and $x < 0$, respectively. Hence f and df/dx are well-defined except at $x = 0$, where we define $f(0) = \mathcal{F}$ for some connected set \mathcal{F} such that $f^\pm(0) \in \mathcal{F}$. It will be useful to write this as

$$f(0) = \mathcal{F} := \{F(0; u) : u \in [-1, +1]\} , \quad (2)$$

for some function $F(x; u)$ that is continuous in x and u , and satisfies $F(x; +1) = f^+(x)$, $F(x; -1) = f^-(x)$. If the right-hand side of (1) is continuous then (2) reduces to a single value $f(0) = \mathcal{F} = F(0; u) = f^\pm(0)$, otherwise $f^+(0) \neq f^-(0)$ and $f(0) = \mathcal{F}$ is a set that contains $f^+(0)$ and $f^-(0)$. Letting $\dot{x} \in \mathcal{F}$ at $x = 0$ turns (1) into a differential inclusion (i.e. a set-valued differential equation, see [7]), permitting continuous solution trajectories to exist for all x .

Let us assume $0 \in \mathcal{F}$; this condition is crucial to permit pausing at $x = 0$.

The standard example, though it is often treated as an isolated case, is the initial value problem

$$\dot{x} = \sqrt{|x|} , \quad x(0) = 0 ,$$

for $x \in \mathbb{R}^+$. Solutions exist because $\sqrt{|x|}$ is continuous, but are non-unique because $\sqrt{|x|}$ is non-differentiable at $x = 0$.

Solutions of (1) are trajectories $x = \phi(t)$ satisfying

$$\frac{d}{dt}\phi(t) = f(\phi(t)) \quad \text{s.t.} \quad \phi(0) = 0. \quad (3)$$

We define a *pausing solution* $\phi = \phi_\tau$ as satisfying, for some $\tau > 0$,

$$\phi_\tau(t) = \begin{cases} \phi_-(t) & \text{if } t < 0, \\ 0 & \text{if } 0 \leq t \leq \tau, \\ \phi_+(t - \tau) & \text{if } t > \tau, \end{cases} \quad (4)$$

where ϕ_\pm are continuous functions such that $\phi_\pm(0) = 0$, and $\phi_-(-d) < 0 < \phi_+(+d)$ for any (small) $d > 0$. The time interval $t \in [0, \tau]$ corresponds to an arbitrarily long pause in motion at $x = 0$, and the notation ϕ_τ emphasises that pausing solutions are a family of orbits parameterized by their exit time τ from the non-differentiable point $x = 0$.

The following result applies if f is not continuously differentiable.

Theorem 1. *Suppose that $\dot{x} = f(x)$ with $x \in \mathbb{R}$ and $0 \in f(0)$. If there exist solutions $\phi_+(t)$ for $t \geq 0$ such that $\phi_+(0) = 0$ and $\phi_+(t)$ is not identically zero, and $\phi_-(t)$ for $t \leq 0$ such that $\phi_-(0) = 0$, then for all $\tau \in \mathbb{R}^+$ the functions ϕ_τ defined by (4) are also solutions.*

The proof is by direct verification that $x(t) = \phi_\tau(t)$ satisfies the differential equation $\dot{x} = f(x)$. Of course, since $x = 0$ is by definition a solution, the existence of the non-trivial solution $\phi(t)$ with $\phi(0) = 0$ implies that f is not smooth enough to apply any of the standard uniqueness theorems concerning solutions of differential equations. As we shall see in the following sections, in the context of piecewise-smooth systems this constraint is less onerous.

In this paper we attempt to resolve the time ambiguity at $x = 0$ in two stages. The first is to consider a small interval $[-\varepsilon, +\varepsilon]$ around $x = 0$, on which we derive a regular dynamical system by re-scaling coordinates, and subsequently let $\varepsilon \rightarrow 0$. The second step is an asymptotic balance to show that the time τ taken to pass through the interval $x \in [-\varepsilon, +\varepsilon]$ can remain finite as we let $\varepsilon \rightarrow 0$.

This device of singular re-scalings, commonly used in singular perturbation and asymptotics (see e.g. [2, 10, 16, 22, 26, 25]), has become known as *blowing-up* a singular point $x = 0$ (see e.g. [17, 20]) or introducing a *switching layer* at a point of discontinuity $x = 0$ (see [13, 15]).

In the first step, the small blow-up interval $[-\varepsilon, +\varepsilon]$ is mapped to some $u \in [-1, +1]$ via a diffeomorphism $x = \varepsilon\chi(u, \varepsilon)$ satisfying

$\chi(\pm 1, \varepsilon) = \pm 1$ (the simplest example being $x = \varepsilon u$). In the u -coordinates we obtain from (1) an equation

$$\varepsilon \dot{u} = g(u, \varepsilon), \quad u \in (-1, +1), \quad (5)$$

for some function $g(u, \varepsilon)$. The object is to choose χ so that (5) gives well-defined dynamics on the interval, and hence resolves the dynamics at $x = 0$ in a manner consistent with (1)-(4). Directly applying the mapping $x \mapsto u$ to (1) gives the right-hand side of (5) as simply $g(u, \varepsilon) = f(\chi(u, \varepsilon))$ if f is continuous. If f is discontinuous at $x = 0$ then we use (2) to express $f(0)$, and we can then choose the u in the mapping $\chi(u, \varepsilon)$ to be the same variable appearing in the function $F(x; u)$, to obtain $g(u, \varepsilon) = F(\chi(u, \varepsilon); u)$.

Although we can obtain a well-defined system (5) through such a re-scaling (1), we must be aware that the result is not unique. The system (5) can contain additional terms and still remain consistent with (1), in the form of functions $h(u, \varepsilon)$ that satisfy $h(\pm 1, \varepsilon) = 0$, i.e. which vanish at the boundary of the blow-up interval. These are called *hidden terms* [13, 15], and the behaviours they induce are called *hidden dynamics*. With hidden terms the righthand side of (5) takes the form $g(u, \varepsilon) = f(\chi(u, \varepsilon)) + h(u, \varepsilon)$ if f is continuous, and $g(u, \varepsilon) = F(\chi(u, \varepsilon); u) + h(u, \varepsilon)$ if f is discontinuous. We shall see several examples throughout the following sections, showing how they allow the different possible behaviours at $x = 0$ to be expressed explicitly.

The time τ taken to pass through the blow-up interval $u \in [-1, +1]$, and thereby from $x < 0$ to $x > 0$, can then be calculated. For certain parameters this ‘pausing’ time is typically found to be zero, for others it is infinite. In between, for a particular balance between ε and the parameters of the system, the pausing time takes the form

$$\tau = B + \mathcal{O}(\varepsilon) \quad (6)$$

for any arbitrary $B > 0$ whose limit is finite and ε independent, i.e. $\varepsilon/B \rightarrow 0$ and $\varepsilon B \rightarrow 0$ as $\varepsilon \rightarrow 0$.

The balancing is done by finding a quantity ξ in the system with a special value ξ_* , at which (6) can be shown to hold by approaching $\xi = \xi_*$ along certain values $\xi = \xi(B, \varepsilon)$, where B is any positive real number. The quantity ξ is related to constants appearing in f or related to an initial condition $x(0)$. The typical mechanism behind this is depicted schematically in fig. 1. For $\xi > \xi_*$ there exists a (possibly degenerate) equilibrium of the u dynamics, a point $u_* \in (-1, +1)$ where $g(u_*, \varepsilon) = 0$. Solutions intersect the equilibrium and cannot pass through the interval, so they become stuck at $x = 0$. For $\xi \lesssim \xi_*$ the equilibrium misses the equilibrium (because it either ceases to exist, or lies outside

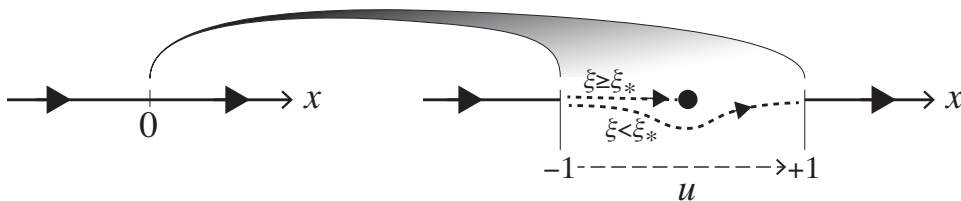


FIGURE 1. A schematic depiction of blow-up (from the left picture to the right picture) and arbitrarily long pausing (as blown up trajectories feel arbitrary slowing for ξ close to ξ_*).

the solution trajectory in higher dimensions). Despite this the variable u slows as it passes close to u_* , creating a delay anywhere on the interval $[0, \infty)$ for ξ close enough to ξ_* . For an appropriate definition of ξ in terms of B this slowing of u remains finite as the region $x \in [-\varepsilon, +\varepsilon]$ collapses to $x = 0$ with $\varepsilon \rightarrow 0$.

For this to work, and thus for arbitrary pausing to occur, the equilibrium u_* of the blow-up system must have both inward and outward trajectories, and in the examples we study this takes the form either of a saddle or saddle-node.

3. THE CLASSIC EXAMPLE OF INDETERMINACY IN CONTINUOUS SYSTEMS

We start with an example often used to show that continuity is not sufficient to guarantee uniqueness of solutions of an initial value problem. The non-uniqueness can be resolved by a re-scaling of variables, except near a particular parameter value for which arbitrarily long pausing occurs.

Consider the scalar system

$$\dot{x} = |x|^\gamma, \quad x(0) = 0, \quad 0 < \gamma \leq 1. \quad (7)$$

The classic example is often quoted with $\gamma = 1/2$, but the problem changes somewhat interestingly for different γ values. The example is not without application, such as in [3], where a particle following vertical motion along a curve $y = |x|^\gamma$ is revealed to exhibit behaviour similar to that shown below for $1/2 < \gamma \leq 1$.

For all $\tau \geq 0$ the function

$$\phi_\tau(t) = \begin{cases} -(1-\gamma)^{\frac{1}{1-\gamma}} |t|^{\frac{1}{1-\gamma}} & \text{if } t < 0, \\ 0 & \text{if } 0 \leq t \leq \tau, \\ +(1-\gamma)^{\frac{1}{1-\gamma}} |t-\tau|^{\frac{1}{1-\gamma}} & \text{if } t > \tau, \end{cases} \quad (8)$$

is a solution of (7), as is verified by direct substitution of $x(t) = \phi_\tau(t)$. Note that this is valid for all $0 < \gamma \leq 1$. From this we can construct a solution that passes from some $x = a < 0$ to some $x = b > 0$, but the time taken to travel from a to b is non-unique. The time taken to reach $x(0) = 0$ from any $a < 0$ is $|a|^{1-\gamma}/(1-\gamma)$, and the time taken to reach any point $x(\tau + b) > 0$ from $x(\tau) = 0$ is $|x(\tau + b)|^{1-\gamma}/(1-\gamma)$; both are uniquely determined by a and b . These are connected through the point $x = 0$ itself, however, where a solution can remain fixed for any real-valued time τ .

Note that as $\gamma \rightarrow 1$ the times to reach/depart $x = 0$ tend to infinity, and at $\gamma = 1$ only $\phi_\infty(t)$ is a solution of (7), meaning a solution at $x = 0$ will remain there for all time. At the other extreme in the limit $\gamma \rightarrow 0$, only $\phi_0(t)$ is a solution of (7), and solutions cross through $x = 0$ in zero time. For $0 < \gamma \leq 1$ it appears from (8) that the time to cross $x = 0$ could take any value between these extremes.

We can attempt to distinguish between solutions that pause for different times at $x = 0$ using the method outlined in section 2. We will show the following.

Lemma 2. *The point $x = 0$ in (7) maps onto a well-defined dynamical system on the interval $u \in [-1, +1]$ via a transformation $x = \varepsilon|u|^{1/(1-\gamma)} \text{sign}(u)$ for $\varepsilon \rightarrow 0$. Then for any $\tau \geq 0$ there exist solutions of (7) that traverse $x = 0$, but pause there for a time τ . For $\gamma < 1$, pausing for a time $\tau > 0$ is possible only in the presence of hidden terms.*

We prove this constructively by finding the time taken for solutions of (12) to evolve through the point $x = 0$ when blown up into the interval $u \in [-1, +1]$.

To study the small interval $x \in [-\varepsilon, +\varepsilon]$ for $\varepsilon \rightarrow 0$, take the transformation given in the lemma, $x = \varepsilon|u|^{1/(1-\gamma)} \text{sign}(u)$ with $u \in [-1, +1]$ and $\varepsilon > 0$. The equation (7) transforms to $|u|^{1/(1-\gamma)} \dot{u} = |u|^{1/(1-\gamma)} (1-\gamma) \varepsilon^{\gamma-1}$. This admits a static solution $u(t) \equiv 0$, which we can discard since it does not touch the boundaries of the region $u \in [-1, +1]$ and hence cannot be part of the solution (8). It also admits a dynamic solution that satisfies

$$\dot{u} = (1-\gamma) \varepsilon^{\gamma-1}, \quad u \in (-1, +1), \quad (9)$$

the right-hand side of which is constant. This result is unique in the sense that it is the only transformation of the form $x = \varepsilon|u|^p \text{sign}(u)$ that gives a well-defined dynamical system for u , as a larger power p would create a singularity at $u = 0$, while a smaller power p would give a non-differentiable problem similar to (7). The result is not unique,

and we express this non-uniqueness by introducing hidden terms as discussed in section 2, by writing

$$\dot{u} = (1 - \gamma)\varepsilon^{\gamma-1} + h(u, \varepsilon), \quad u \in (-1, +1), \quad (10)$$

where h is continuous and bounded on $u \in [-1, +1]$, and

$$h(\pm 1, \varepsilon) = 0. \quad (11)$$

The equation (10) represents a family of blow-ups of the system (7) at $x = 0$, with (11) ensuring that the hidden term h vanishes at $u = \pm 1$, so that upon taking $\varepsilon \rightarrow 0$ we obtain (7). For an illustrative example let us consider

$$\dot{u} = (1 - \gamma)\varepsilon^{\gamma-1} \{1 + \alpha(u^2 - 1)\}, \quad u \in (-1, +1), \quad (12)$$

for some arbitrary real constant α . (The function $u^2 - 1$ represents the simplest form of hidden term, and in fact a general model for hidden term is $(u^2 - 1)k(u, \varepsilon)$ for any finite function k , see [15]).

We have then to calculate the time taken to cross the interval $u \in [-1, +1]$. We will first look at what happens in the absence of hidden terms, for which it is easy to show that pausing is only possible at the limiting value $\gamma = 1$, and then we consider the more general case.

3.1. Case 1: a simple blow-up for $\alpha = 0$. If $\alpha = 0$ then (12) reduces to (9), which has unique solutions $u(t) = -1 + (1 - \gamma)\varepsilon^{\gamma-1}t$. These cross from $u = -1$ to $u = +1$ in a time $\tau = 2\varepsilon^{1-\gamma}/(1 - \gamma)$, hence $\tau = \mathcal{O}(\varepsilon^{1-\gamma})$. For $0 < \gamma < 1$ this means $\tau \rightarrow 0$ as $\varepsilon \rightarrow 0$, implying that $x = [-\varepsilon, +\varepsilon] \rightarrow 0$ is traversed in time $\tau = 0$.

This argument assumes that $\varepsilon^{1-\gamma}$ tends to zero with ε , which fails if $\gamma = 1$. By taking $\varepsilon \rightarrow 0$ and $\gamma \rightarrow 1$ appropriately we will show that the transition time takes every value $\tau = B \in (0, \infty)$ at this limit. This involves balancing the smallness of ε and of $\delta\gamma = 1 - \gamma$ that appear in the time $\tau = 2\varepsilon^{1-\gamma}/(1 - \gamma)$, such that

$$\tau = \frac{2}{\delta\gamma}\varepsilon^{\delta\gamma} = B \quad (13)$$

for any $B \in \mathbb{R}^+$. Firstly note that solutions of (13) satisfy $1/\delta\gamma + \log \varepsilon < B < 1/\delta\gamma$. Let $\mu = -\delta\gamma \log \varepsilon$, then the required balance is obtained if

$$\mu + \log \mu = \log \frac{2 \log \varepsilon}{-B} \quad \text{or} \quad \mu e^\mu = \frac{2 \log \varepsilon}{-B}, \quad (14)$$

whose solution is a Lambert W-function [1], with series expansion for large argument

$$\mu = W_L\left(\frac{2 \log \varepsilon}{-B}\right) \approx \nu - \log \nu + \frac{\log \nu}{\nu} + \frac{(\log \nu - 2) \log \nu}{2\nu^2} + \dots, \quad (15)$$

where $\nu = \log \frac{2 \log \varepsilon}{-B}$, hence

$$\gamma \approx 1 + \frac{\nu}{\log \varepsilon} - \frac{\log \nu}{\log \varepsilon} \left\{ 1 - \frac{1}{\nu} - \frac{(\log \nu - 2)}{2\nu^2} + \dots \right\}. \quad (16)$$

The quantities μ and ν are both large for very small ε , but smaller than $-\log \varepsilon$, ensuring $\delta\gamma \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus for $\gamma = 1$ we can always find a continuous limit satisfying (14) (the solutions of which we can obtain asymptotically from (15)-(16)), such that we obtain arbitrary pausing times $\tau = B$. \square

3.2. Case 2: the blow-up with hidden terms, $\alpha \neq 0$. Integrating (12) for the transition time τ across $u \in [-1, +1]$ gives

$$\tau = \frac{2\varepsilon^{1-\gamma}}{(1-\gamma)\rho} \arctan(1/\rho) = \frac{\varepsilon^{1-\gamma}}{(1-\gamma)i\rho} \log \frac{\rho+i}{\rho-i}, \quad (17)$$

where $\rho = \sqrt{\frac{1}{\alpha} - 1}$. For $\alpha > 1$ we have $\tau \rightarrow \infty$, so solutions become stuck inside the interval for all time, and hence cannot escape from $x = 0$. For $\alpha < 1$ we have $\tau = \mathcal{O}(\varepsilon^{1-\gamma})$, so letting $\varepsilon \rightarrow 0$ we find that solutions cross the interval in zero time.

This fails at $\alpha = 1$ (or at $\gamma = 1$ which we dealt with in Case 1). The jump from $\tau = 0$ for $\alpha < 1$ to $\tau \rightarrow \infty$ for $\alpha > 1$ suggests that at $\alpha = 1$ the transition time passes through all values on \mathbb{R}^+ . To show this explicitly, the appropriate balance that provides finite pausing at $\alpha = 1$ is given by

$$\alpha = 1 - \varepsilon^{2(1-\gamma)} \pi^2 / (1-\gamma)^2 B^2 \quad (18)$$

for some arbitrary positive constant B . Substituting this into (17), then expanding for small ε , gives

$$\tau = B - \frac{2\varepsilon^{1-\gamma}}{1-\gamma} + \mathcal{O}(\varepsilon^{2(1-\gamma)}), \quad (19)$$

and hence $\tau \rightarrow B$ as $\varepsilon \rightarrow 0$. \square

Thus in both cases solutions cross $x = [-\varepsilon, +\varepsilon] \rightarrow 0$ in zero time for some values of γ and α , but at certain values their speed through the interval becomes slow enough to remain non-vanishing as $\varepsilon \rightarrow 0$. Solutions then pause at $x = 0$ for an arbitrary time B , where B can be found to take any value by balancing the smallness of ε , $1 - \alpha$, and $1 - \gamma$, as they tend to zero.

Without hidden terms this pausing occurs at $\gamma = 1$. For $\gamma < 1$ the system (7) has a finite time attractor at $x = 0$, but its blow-up (9) gives a finite speed of travel through $x = 0$, so there is no pausing as solutions pass from $x < 0$ to $x > 0$. For $\gamma \geq 1$ the system (7) has an

infinite time attractor, which makes passage through $x = 0$ impossible. As γ approaches unity from below, the speed given by (9) slows to zero, leading to arbitrarily long pausing at $\gamma = 1$.

With hidden terms pausing instead depends on α , with no pausing for most values of $\alpha < 1$, when (12) gives a non-vanishing speed of travel across $x = 0$. As α approaches unity, a bifurcation gives rise to a pair of equilibria in the u dynamics (at $u = \pm i\rho$), so that crossing again becomes impossible for $\alpha \geq 1$. As α approaches unity from below, the impending bifurcation slows the dynamics across the interval $u \in [-1, +1]$ and leads to arbitrarily long pausing at $\alpha = 1$.

The final result can be understood in quite simple terms. The problem is to find the time t taken to traverse an interval $-\varepsilon \leq x \leq +\varepsilon$ as $\varepsilon \rightarrow 0$. If we assume finite speed through $x = 0$ this time is $\tau = 2\varepsilon^{1-\gamma}/(1-\gamma)$, directly from (8). The blow-up analysis finds a coordinate scaling in which such a finite speed system exists, and also allows us to explore cases where this is not possible using hidden terms. The introduction of hidden terms is therefore a modeling problem, a means to explicitly express the different possible dynamics through $x = 0$ that a system may exhibit.

In the following sections we show that time indeterminacy like this is common by taking examples motivated by applications, and we show that typically a scaling can be found that gives insight into when the indeterminacy is resolvable (for $0 < \gamma < 1$ and $0 < \alpha < 1$ above), and when arbitrary pausing is possible (when $\gamma = 1$ or $\alpha = 1$ above).

4. PAUSING AT A DISCONTINUITY

The previous example involved a continuous vector field with a discontinuity in its first derivative. Here we consider systems where pausing occurs as a flow crosses a discontinuity in the vector field itself. We consider a discontinuity that occurs either at a hypersurface or an intersection of hypersurfaces.

4.1. Example of a single discontinuity. For any system in which there is a discontinuity along some hypersurface, with solutions entering from one side and departing from the other, we can find conditions where finite time indeterminacy occurs.

Before considering more involved cases, let us look at a very simple discontinuity, taking a system that switches between $\dot{x} = 1$ for $x < 0$ and $\dot{x} = 3$ for $x > 0$, given by

$$\dot{x} = 2 + u + \alpha(u^2 - 1), \quad u = \text{sign}(x). \quad (20)$$

This example contains a hidden term $\alpha(u^2 - 1)$ from the outset, as a simplification of a model of static friction in mechanical oscillators proposed in e.g. [13, 14].

At $x = 0$ we let u take all values on the interval $[-1, +1]$, giving the right-hand side of (20) as in (2). For $\tau \geq 0$ the function

$$\phi_\tau(t) = \begin{cases} t & \text{if } t < 0, \\ 0 & \text{if } 0 \leq t \leq \tau, \\ 3(t - \tau) & \text{if } t > \tau, \end{cases} \quad (21)$$

is a solution $x(t) = \phi_\tau(t)$ for any τ , as is verified by substituting (21) back into (20), noting that the point 0 lies in the right-hand side of (20) at $x = 0$ given $u \in [-1, +1]$.

When considering the interval $x \in [-\varepsilon, +\varepsilon]$ for $\varepsilon \rightarrow 0$ we can take u as our blow-up variable, letting $x = \varepsilon u$ on $u \in [-1, +1]$. In this context the interval $u \in [-1, +1]$ is called a *switching layer* [13]. We can substitute $x = \varepsilon u$ directly into (20), but should also account for the non-uniqueness of the blow-up by introducing a hidden term. For simplicity we will again take the simplest example as some factor of $u^2 - 1$. This gives the dynamics at $x = 0$ as

$$\varepsilon \dot{u} = 2 + u + \alpha(u^2 - 1), \quad u \in (-1, +1), \quad (22)$$

for $\varepsilon \rightarrow 0$ and some constant $\alpha \geq 0$.

Lemma 3. *For any real $\tau \geq 0$, given the blow-up of (20) at $x = 0$ into the system (22), solutions of (20) traverse $x = 0$ in zero time when $\alpha < \alpha_c$ and become stuck at $x = 0$ for $\alpha > \alpha_c$, where $\alpha_c = 1 + \sqrt{3}/2$. For $\alpha = \alpha_c$ solutions pause at $x = 0$ for any arbitrary time τ at $x = 0$. If we take $\alpha = \alpha_c - \varepsilon^2 \pi^2 / \sqrt{3} B^2$ for any $B \in \mathbb{R}^+$, then the transition time in the limit $\varepsilon \rightarrow 0$ is $\tau = B$.*

To show this consider the dynamics of u on the interval $(-1, +1)$, defining $\alpha_c = 1 + \sqrt{3}/2$ as stated in the lemma.

For $\alpha > \alpha_c$ there exist two equilibria of (22), at

$$u = u_\pm := \frac{1}{2\alpha} \left(-1 \pm \sqrt{4\alpha^2 - 8\alpha + 1} \right), \quad (23)$$

one attracting and one repelling. Thus when a solution reaches $x = 0$, the quantity u evolves to the attractor u_- , and remains there for all time, so solutions cannot cross the interval $u \in (-1, +1)$, and the time taken to traverse $x = 0$ is $\tau \rightarrow \infty$.

For $\alpha < \alpha_c$, the right-hand side of (22) is non-vanishing on $u \in (-1, +1)$, therefore solutions evolve across the interval in time $\tau = \mathcal{O}(\varepsilon) \rightarrow 0$.

We must therefore look closer at how the time τ jumps from zero to infinity as α tends to α_c from below. For $\alpha < \alpha_c$, solutions $u(t)$ of (22) with initial condition $u(0) = -1$ are given by

$$u(t) = -\frac{1}{2\alpha} + \frac{\rho}{2\alpha} \tan\left(\frac{\rho t}{2\varepsilon} + \arctan\left(\frac{1-2\alpha}{\rho}\right)\right), \quad (24)$$

with $\rho = \sqrt{-4\alpha^2 + 8\alpha - 1}$ or, in terms of α_c , $\rho = 2\sqrt{\alpha_c - \alpha}\sqrt{\sqrt{3} + \alpha - \alpha_c}$. At time $t = \tau$ the solution has crossed the discontinuity and reached $u(\tau) = +1$. Then (24) gives

$$+1 = -\frac{1}{2\alpha} + \frac{\rho}{2\alpha} \tan\left(\frac{\rho\tau}{2\varepsilon} + \arctan\left(\frac{1-2\alpha}{\rho}\right)\right).$$

Re-arranging this to

$$\frac{\rho\tau}{2\varepsilon} = \arctan\frac{1+2\alpha}{\rho} - \arctan\frac{1-2\alpha}{\rho},$$

then using double angle formulae gives

$$\tan\frac{\rho\tau}{2\varepsilon} = \tan\left(\arctan\frac{1+2\alpha}{\rho} - \arctan\frac{1-2\alpha}{\rho}\right) = \frac{\rho}{2(1-\alpha)},$$

and hence

$$\tau = \frac{2\varepsilon}{\rho} \arctan\frac{\rho}{2(1-\alpha)} + \frac{2\pi\varepsilon}{\rho} H(\alpha - 1), \quad (25)$$

where H is the Heaviside function (this appears when inverting the tan function in such a way as to guarantee continuity of τ with α).

Thus $\tau \rightarrow 0$ as $\varepsilon \rightarrow 0$, provided $\rho = 2\sqrt{\alpha_c - \alpha}\sqrt{\sqrt{3} + \alpha - \alpha_c}$ is not too small, i.e. α is not too close to α_c . If we take the limit towards $\alpha = \alpha_c$ with the correct balance of $\alpha - \alpha_c$ and ε , however, (25) can attain any real positive value. The appropriate scaling is given by $\alpha = \alpha_c - \varepsilon^2\pi^2/\sqrt{3}B^2$ for any $B > 0$, which substituted into (25), with a little algebra, gives

$$\begin{aligned} \tau &= \frac{B}{\pi\sqrt{1+\varepsilon^2\pi^2/3B^2}} \arctan\left(\frac{\varepsilon\pi\sqrt{1+\varepsilon^2\pi^2/3B^2}}{B(1-\alpha)}\right) + \frac{BH(\alpha-1)}{\sqrt{1+\varepsilon^2\pi^2/3B^2}} \\ &= BH(\alpha-1) + \frac{B}{\pi} \arctan\left(\frac{\varepsilon\pi}{B(1-\alpha)}\right) + O(\varepsilon^2) \\ &= BH(\alpha-1) + O(\varepsilon). \end{aligned} \quad (26)$$

(We include the first two rows to show how the expansion in ε arises). Thus for $\alpha = \alpha_c$, noting $H(\alpha-1) = 1$ since $\alpha = \alpha_c - \varepsilon^2\pi^2/3B^2 > 1$, as $\varepsilon \rightarrow 0$ we are left with a transition time $\tau = B$ for any $B > 0$, i.e. the time τ taken to cross $x = 0$ is any arbitrary value $\tau = B$. For $\alpha < \alpha_c$ we obtain uniquely $\tau = O(\varepsilon) \rightarrow 0$. \square

Thus solutions cannot cross the discontinuity for $\alpha > \alpha_c$, while in the range $0 < \alpha < \alpha_c$ solutions cross the discontinuity in zero time. By considering $\alpha = \alpha_c$ to be approached by a limit remaining order ε^2 close to α_c from below, we find solutions that take an arbitrary time

to cross the discontinuity at $x = 0$, since the pausing time approaches an arbitrary $\tau = B$ as $\varepsilon \rightarrow 0$.

4.2. Example of a double discontinuity from genetic regulation. The example above can be extended quite directly to consider higher dimensional vector fields with a discontinuity along some hypersurface. Let us therefore turn to the problem of discontinuities that occur along two hypersurfaces that intersect in two or more dimensions.

Here we look at an example derived from models of protein production in genetic regulation, which typically involve multiple switches (one per gene). These models describe the variation of protein concentrations, say X and Y , as production is turned on or off by the associated genes at threshold values θ_X and θ_Y . The switches are usually approximated as sigmoidal Hill functions [9], $Z(X)$ and $Z(Y)$, where $Z(X) = X^p / (X^p + \theta_X^p) \xrightarrow{p \rightarrow \infty} \frac{1}{2} + \frac{1}{2} \text{sign}(X - \theta_X)$, and models take the form

$$\begin{aligned}\dot{X} &= \mathcal{B}_X(Z(X), Z(Y)) - c_X X, \\ \dot{Y} &= \mathcal{B}_Y(Z(X), Z(Y)) - c_Y Y,\end{aligned}$$

where the c_i are positive constants, and the \mathcal{B}_i are multilinear functions representing Boolean expressions of gene interactions.

The discontinuities here occur at $X = \theta_X$ and $Y = \theta_Y$, so let us introduce variables $x = X - \theta_X$ and $y = Y - \theta_Y$. Examples were studied in [21, 11] where a saddlepoint, or a saddle and a node, were observed to play a crucial part in pausing as trajectories evolved through the point $x = y = 0$. In those papers the switches Z were treated as smooth functions, but we will show here that pausing can be revealed easily by taking the switches to be discontinuous and using the analysis from section 2.

The decay terms $-c_X X$ and $-c_Y Y$ do not qualitatively alter our results near $X = Y = 0$, so we shall omit them for brevity and concentrate on the role of the switches (more precisely these linear terms constitute a *regular* perturbation of the model, which add terms of order ε to our calculations when we blow up, and these do not affect the qualitative result). We can illustrate the mechanism of pausing with the system

$$\dot{x} = -u_x, \quad \dot{y} = \gamma + u_y + 2u_x u_y, \quad (27)$$

where $u_x = \text{sign}(x)$ and $u_y = \text{sign}(y)$, for $|\gamma| < 1$. This is based on examples in [21, 11]. The system is piecewise constant, with vector fields pointing in towards the line $x = 0$ and the half-line $x < 0 = y$, and outwards from the half-line $x > 0 = y$, as depicted in fig. 2.

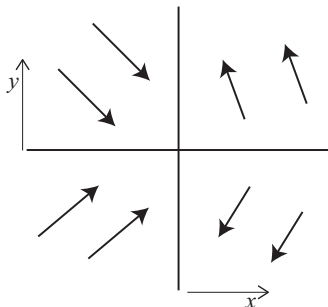


FIGURE 2. Discontinuous vector fields of the double switching system.

Let us consider a solution that begins in the upper left quadrant, $x < 0 < y$, and hits the discontinuity half-line $x < 0 = y$ at time $t = -c$ at a point $(-c, 0)$ for $c > 0$. The trajectory

$$\phi_{\tau}^{\pm}(t) = \begin{cases} (t, (\gamma - 1)(t + c)) & \text{if } t < -c, \\ (t, 0) & \text{if } -c < t < 0, \\ (0, 0) & \text{if } 0 < t < \tau, \\ (0, (\gamma \pm 1)(t - \tau)) & \text{if } t > \tau, \end{cases} \quad (28)$$

is a solution to (27) for any τ . The \pm on the right-hand side of (28) denotes a spatial ambiguity, namely that solutions may exit along $x = 0$ into either $y > 0$ or $y < 0$.

After reaching the discontinuity at $(-c, 0)$, the only possible onward evolution is along the half-line $y = 0$ in $x < 0$, until reaching the origin $(0, 0)$ where the switching surfaces $x = 0$ and $y = 0$ intersect. At this point, the solution may remain fixed at the origin, or at any time τ it may leave along $x = 0$ in $y > 0$ or $y < 0$.

As before, we shall seek the time τ for which the solution pauses at the origin, and also, in this case, resolve the spatial determinacy of whether the solution exits along the y positive or negative branch of $x = 0$ indicated by the \pm sign in (28).

Lemma 4. *For any $\tau > 0$, if $\gamma = 0$ then there exist solutions of (27) that traverse $x = y = 0$, pausing for a time τ as they do so. Taking $\gamma = \gamma_{\infty} + e^{-B/\varepsilon}$ for any $B \in \mathbb{R}^+$, where $\gamma_{\infty} = 2/(2 - (1 + e^2)e^{c/\varepsilon})$ in terms of a non-negative infinitesimal ε , then the transition time in the limit $\varepsilon \rightarrow 0$ is $\tau = B$.*

To show this we must look at the solution's evolution inside the discontinuity surface $x < y = 0$, and then inside the intersection of discontinuities at $x = y = 0$. As in section 4.1 we can use the discontinuous term u itself to blow up the non-differentiable point $x = 0$,

except here we have two discontinuous terms u_x and u_y , which we use to blow-up the lines $x = 0$ and $y = 0$ respectively.

At $t = -c$ the solution lies at $(-c, 0)$, where a discontinuity takes place in the y -direction as u_y switches between ± 1 . The blow-up of (27) on $y = 0$ is performed by letting $y = \varepsilon_y u_y$ for $u_y \in [-1, +1]$ and $\varepsilon_y \rightarrow 0$, giving $\varepsilon_y \dot{u}_y = \gamma + u_y + 2u_x u_y$. Combined with the \dot{x} equation this gives, on $x < 0 = y$,

$$(\dot{x}, \varepsilon_y \dot{u}_y) = (1, \gamma - u_y) \quad : \quad u_y \in (-1, 1). \quad (29)$$

This is illustrated in fig. 3. The solution lies at $x = -c$, $u_y = +1$, at time $t = -c$. For $-c < t < 0$ the solution of (29) gives $x(t) = t$ and $u_y(t) = \gamma + (1 - \gamma)e^{-(t+c)/\varepsilon_y}$, and at time $t = 0$ it arrives at $x(0) = 0$ and $u_y(0) = \gamma + (1 - \gamma)e^{-c/\varepsilon_y}$. For large c or in the limit $\varepsilon_y \rightarrow 0$ we have $u_y(0) \rightarrow \gamma$.

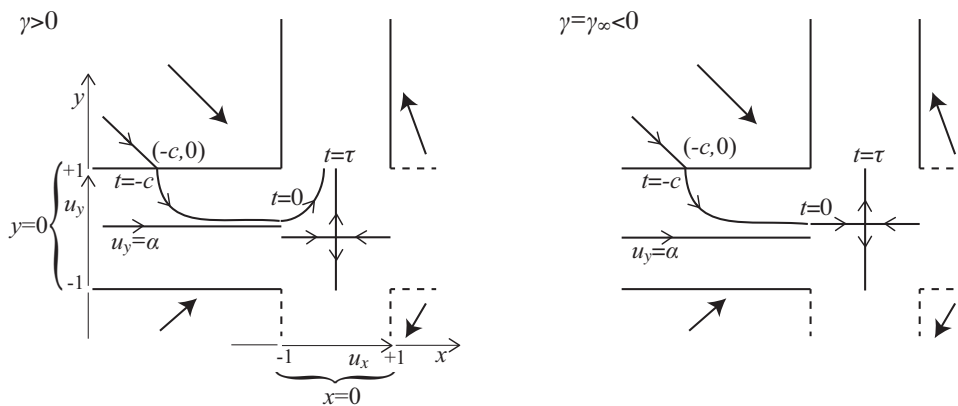


FIGURE 3. Geometry of the double switch, showing a solution arriving at $y = 0$, evolving through the interval $u_y \in (-1, +1)$, then through the double-interval $(u_x, u_y) \in (-1, +1) \times (-1, +1)$ in which there is a saddlepoint, and exiting into $y > 0$ at $t = \tau$.

At time $t = 0$ the solution now lies at $x = y = 0$, where both u_x and u_y switch between values ± 1 . Each transition is described by a blow-up system, letting $x = \varepsilon_x u_x$ and $y = \varepsilon_y u_y$ on $(u_x, u_y) \in [-1, +1]^2$, giving $\varepsilon_x \dot{u}_x = -u_x$ and $\varepsilon_y \dot{u}_y = \gamma + u_y + 2u_x u_y$ for non-negative infinitesimals $\varepsilon_x, \varepsilon_y$. For simplicity we can let $\varepsilon = \varepsilon_x = \varepsilon_y$ without loss of generality, hence on $x = y = 0$ we have

$$(\varepsilon \dot{u}_x, \varepsilon \dot{u}_y) = (-u_x, \gamma + u_y + 2u_x u_y), \quad (30)$$

for $u_x, u_y \in (-1, 1)$ and $\varepsilon \rightarrow 0$, as illustrated in fig. 3. This system has a saddlepoint at $u_x = 0$, $u_y = -\gamma$, whose stable manifold lies

along $2u_x u_y = \gamma(e^{-2u_x} - 1)$, intersecting the coordinate $u_x = -1$ at $u_y = u_y^{sep} := \frac{1}{2}\gamma(1 - e^2)$.

With initial condition $u_x(0) = -1$ and $u_y(0) = \gamma + (1 - \gamma)e^{-c/\varepsilon}$ at time $t = 0$, solutions take the form

$$\begin{aligned} u_x(t) &= -e^{-t/\varepsilon}, \\ u_y(t) &= \frac{1}{2u_x(t)} \left\{ -\gamma + e^{-2(u_x(t)+1)} (\gamma - 2u_y(0)) \right\}. \end{aligned} \quad (31)$$

Hence for any $t > 0$ as $\varepsilon \rightarrow 0$, we have $u_x \rightarrow 0$ and $u_y \rightarrow \gamma\infty$, meaning $u_y(t)$ will reach $u_y(t) = \text{sign}(\gamma)$ in zero time. Passage through $x = y = 0$ therefore occurs instantaneously, with exit into $y > 0$ if $\gamma > 0$ and into $y < 0$ if $\gamma < 0$ at time $t = \tau$.

This resolves the spatial indeterminacy in (28), as solutions are given by $\phi_0^{\text{sign}\gamma}(t)$ for $\gamma \neq 0$, and $u_y \xrightarrow{\varepsilon \rightarrow 0} \gamma\infty$ (cutting off at $u_y = \text{sign}(\gamma)$) provided $\gamma \neq 0$. If $\gamma = 0$ then the solution is still indeterminate, and we must treat (31) more carefully before we let $\varepsilon \rightarrow 0$.

In terms of ε , there are particular values of γ and c for which $\tau \rightarrow \infty$ (even for $\varepsilon > 0$), hence the solution sticks to $x = y = 0$ for all time, as shown in the right picture in fig. 3. Let us denote this value of γ by γ_∞ , where (31) gives

$$\begin{aligned} 0 &= \gamma_\infty + e^{-2} (\gamma_\infty + 2(1 - \gamma_\infty)e^{-c/\varepsilon}) \\ \Rightarrow e^{-c/\varepsilon} &= \frac{\gamma_\infty(1 + e^2)}{2(\gamma_\infty - 1)}. \end{aligned} \quad (32)$$

Note that $\gamma_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then consider $\gamma = \gamma_\infty$ to be approached from below. The appropriate scaling to achieve asymptotic balance turns out to be $\gamma = \gamma_\infty \pm e^{-B/\varepsilon}$ for some arbitrary constant $B > 0$, the signs corresponding to solutions that reach $u_y(\tau) = \pm 1$. Substituting $u_y(\tau) = \pm 1$ into (31) for $u_y(t)$ we have

$$\pm 2e^{-\tau/\varepsilon} = \gamma + e^{2(e^{-\tau/\varepsilon}-1)} (\gamma + 2(1 - \gamma)e^{-c/\varepsilon}) \quad (33)$$

and using (32) to eliminate the lone $e^{-c/\varepsilon}$ term, and introducing $\gamma = \gamma_\infty \pm e^{-B/\varepsilon}$, this rearranges to

$$\pm 2e^{-\tau/\varepsilon} = \gamma_\infty \pm e^{-B/\varepsilon} - e^{2(e^{-\tau/\varepsilon}-1)} \left(\gamma_\infty e^2 \pm e^{-B/\varepsilon} \frac{\gamma_\infty e^2 + 1}{\gamma_\infty - 1} \right). \quad (34)$$

Taking a series expansion of the term $e^{2e^{-\tau/\varepsilon}}$ for small $e^{-\tau/\varepsilon}$ gives

$$2(\gamma_\infty \pm 1)e^{-\tau/\varepsilon} = \pm \frac{1 + e^{-2}}{1 - \gamma_\infty} e^{-B/\varepsilon} + \mathcal{O}(e^{-2\tau/\varepsilon}, e^{-\tau/\varepsilon - B/\varepsilon})$$

or, assuming $\tau = \mathcal{O}(B)$, the error term simplifies to

$$2(\gamma_\infty \pm 1)e^{-\tau/\varepsilon} = \pm \frac{1 + e^{-2}}{1 - \gamma_\infty} e^{-B/\varepsilon} + \mathcal{O}(e^{-2B/\varepsilon}) \quad (35)$$

which we can rearrange to find the transition time through the interval $(u_x, u_y) \in [-1, +1] \times [-1, +1]$ as

$$\begin{aligned} \tau &= B - \varepsilon \log \left\{ \frac{1 + e^{-2}}{2(1 \pm \gamma_\infty)(1 - \gamma_\infty)} \right\} + \mathcal{O}(\varepsilon e^{-B/\varepsilon}) \\ &= B + \mathcal{O}(\varepsilon, \varepsilon e^{-B/\varepsilon}) . \end{aligned} \quad (36)$$

Thus in the limit as $\varepsilon \rightarrow 0$ this leaves $\tau \rightarrow B$, giving an arbitrary finite time B of transition through the point $x = y = 0$, i.e. a pausing time $\tau = B$ where B is any positive number. Because $\gamma = \gamma_\infty \pm e^{-B/\varepsilon} \rightarrow \gamma_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$, this indeterminacy arises at $\gamma = 1$, taking the limit with γ exponentially close to zero in ε . \square

In this situation of two switches, solutions become stuck at the intersection for all time only for a special parameter value $\gamma = 0$ (more carefully $\gamma = \gamma_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$). Taking the limit by considering $\gamma = \gamma_\infty \pm e^{-B/\varepsilon}$ where B is any positive value and $\varepsilon \rightarrow 0$, solutions pause for a time $\tau = B$ at $x = y = 0$, after which they evolve along $x = 0$ with $\text{sign}(y) = \text{sign}(\gamma)$.

We have not included hidden terms in this case, looking only at the most simple scenario, but unless they change the qualitative picture (i.e. the existence of a single saddle) inside the layer, then the problem will not be crucially altered. More interesting is to consider the actual problem studied in [21, 11], where pausing occurs as a saddle and node bifurcate inside the layer, hence this involves bringing together the elements of the last two sections: a saddle inside the layer formed by two intersecting switches as in section 4.2, combined with a saddlenode bifurcation as in section 4.1. One can already deduce that zero-time passage will occur in the absence of the saddle and node, or for trajectories who miss the saddle's stable manifold, while arbitrary pausing requires the saddle to exist and affects only trajectories approaching sufficiently close to its stable manifold. Further study of such scenarios are left to future work.

5. PAUSING AT AN IMPACT, AN EXAMPLE FROM MECHANICS

In all of the examples above, a discontinuity occurred at a hypersurface inside the domain of motion, but pausing can also occur when a

discontinuity occurs at the boundary of a domain. The following example is inspired by Painlevé's paradox, which occurs when a rigid rod impacts end-on with a rough (i.e. frictional) surface (see e.g. [23, 5]).

Consider a rod that is in free-flight when the distance from some surface to the rod's nearest endpoint is $x > 0$. An impact happens when $x = 0$, and we want to describe the contact dynamics that then occurs.

A point-like object undergoing an ideal impact would simply bounce off the surface, but a rod may either bounce off, or by rotating it may remain in contact for an interval of time, during which it can either slip along the surface, or stick due to friction. Repeated events of stick-slip or impact chattering may occur. Painlevé's paradox concerns situations in which the outcome of an impact cannot be determined uniquely, see e.g. [23, 4, 5].

We shall see that this ambiguity is actually just another example of those discussed throughout this paper. We give a sketched account that establishes the relation to pausing, directing the reader to the appropriate references for the full analysis, which is somewhat lengthy; that a brief sketch as given here is at all possible is due to certain clarity brought to the problem in [5].

Attempts have been made to resolve the paradox by modeling the impact at $x = 0$ as compliant, so that there is some deformation of the bodies around $x \approx 0$ in contact. A simple way to model this is to let $x = \varepsilon y$ for some non-negative infinitesimal ε , where y represents a virtual displacement or deformation during contact. The corresponding physical displacement is then $x \rightarrow 0$ as $\varepsilon \rightarrow 0$, and we see that this is a form of blow-up of $x = 0$.

The speed during the impact phase is then $v = \varepsilon \dot{y}$. If the acceleration is $\dot{v} = f(y, v)$ for some function f , then the interesting cases are found in [5] to involve the existence of an equilibrium at some $y = -c < 0$, $v = 0$. Then $\dot{v} = f(y, v)$ can be approximated to linear order as

$$\dot{v} = p(y + c) + rv \tag{37}$$

where $p = \frac{\partial}{\partial y} f(-c, 0)$ and $r = \frac{\partial}{\partial v} f(-c, 0)$. These are just the equations of a simple oscillator.

Following [4, 5] we then rescale to some time s and speed u during impact,

$$s = \frac{1}{\sqrt{\varepsilon}} t, \quad u = \frac{dy}{ds} = \sqrt{\varepsilon} \frac{dy}{dt} = \frac{1}{\sqrt{\varepsilon}} v. \tag{38}$$

Letting $q = r\sqrt{\varepsilon}/2$ and denote differentiation with respect to s by a prime, then (37) becomes

$$y' = u, \quad u' = p(y + c) + 2qu. \quad (39)$$

The dynamics inside the impact phase space (y, u) and the free-flight phase space (x, v) are depicted in the main part of fig. 4.

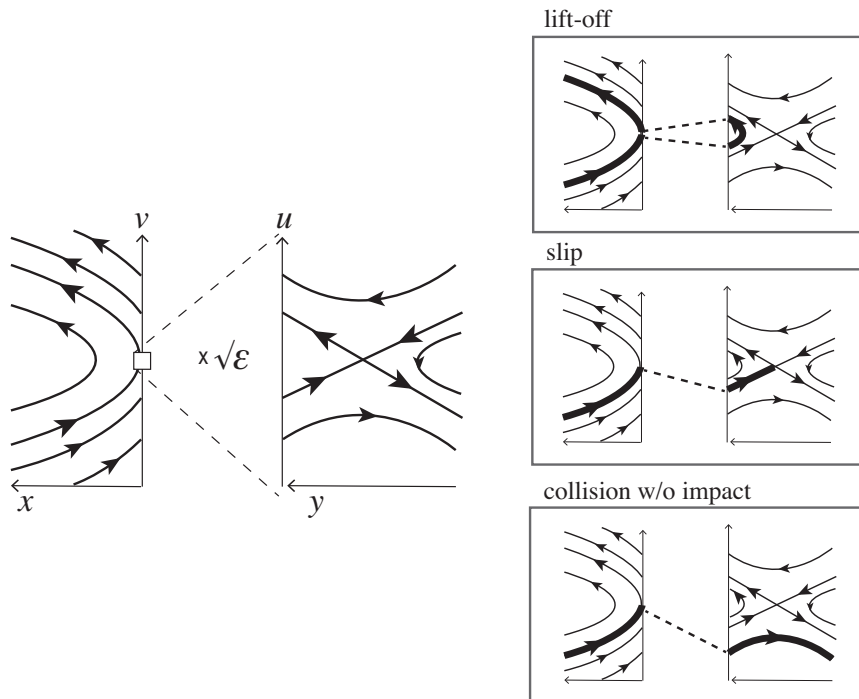


FIGURE 4. Dynamics in the free-flight phase space (x, v) , and the virtual impact phase space (y, u) , blown up by a factor $\sqrt{\varepsilon}$. The three panels on the right show the three different dynamic scenarios revealed by blow-up: slip, lift-off, or so-called collision without impact.

With initial conditions (u_0, y_0) at time $s = 0$, the impact phase has solutions

$$y(s) = -c + e^{qs} \left((y_0 + c) \cosh(\omega s) + \frac{u_0 - q(y_0 + c)}{\omega} \sinh(\omega s) \right) \quad (40)$$

where $\omega = \sqrt{p + q^2}$.

Let us assume $p > 0$, then there is a saddlepoint at $y = c$, $u = 0$, whose stable manifold (the line with gradient $y/u = -(q+\omega)/p$ through the saddle) intersects the impact set $y = 0$ at $u = u_s := -pc/(q + \omega)$.

This schematic description is sufficient to reveal the geometry underlying Painlevé's scenario. If an impact occurs at $x = 0$ with a speed

$v \approx 0$, this may nevertheless correspond to an arbitrary scaled speed u since $v = \sqrt{\varepsilon}u \rightarrow 0$ for any finite u in the rigid impact limit of $\varepsilon \rightarrow 0$.

When the impact begins at $t = 0$, if $u > u_s$ then the solution evolves within the enclosure of the impact surface and the saddle's separatrices, towards $u > 0$ and $y > 0$. The solution then exits towards lift-off in a finite time, giving a real (i.e. unscaled) impact time $t = \mathcal{O}(\sqrt{\varepsilon}) \rightarrow 0$ (top right of fig. 4). If $u < u_s$ then the solution evolves below the saddle deeper into the impact layer, a scenario called "collision-without-impact" (bottom right of fig. 4), in which exit from the impact phase never occurs, hence $y \rightarrow \infty$ as $t \rightarrow \infty$ and there is no lift-off. Between the two, the solution evolves in infinite time towards the saddle, which corresponds to a slipping solution (middle right of fig. 4). (For more detail about the slip, lift-off, and collision without impact scenarios, see [5]).

At $u = u_s$ the impact time t must jump from zero to infinity, and this is where pausing will occur. To calculate the return time $t = \tau$ to $y = 0$, (40) gives with $y_0 = y(\tau) = 0$,

$$0 = -ce^{-q\tau/\sqrt{\varepsilon}} + c \cosh(\omega\tau/\sqrt{\varepsilon}) + \frac{u_0 - qc}{\omega} \sinh(\omega\tau/\sqrt{\varepsilon}). \quad (41)$$

The asymptotic balance to obtain arbitrary pausing in this case is obtained by considering an initial impact velocity scaling as $u_0 = \frac{-pc}{q+\omega} + \frac{e^{-qB/\sqrt{\varepsilon}} - e^{-\omega B/\sqrt{\varepsilon}}}{\sinh(\omega B/\sqrt{\varepsilon})} \omega c$ for some arbitrary B , as substituting this in to (41) we obtain, after a little algebra,

$$e^{-q\tau/\sqrt{\varepsilon}} - \cosh(\omega\tau/\sqrt{\varepsilon}) = e^{-qB/\sqrt{\varepsilon}} - \cosh(\omega B/\sqrt{\varepsilon}), \quad (42)$$

the lefthand side of which is monotonic in τ , implying a unique root, which corresponds to an arbitrary impact time of $\tau = B$.

Similar to the genetic regulation example in section 4.2, pausing here occurs when a trajectory approaches close to the stable manifold to a saddle in the blow-up layer, here in the impact phase. In this case we find that any arbitrary pausing time $\tau = B$ can be found for suitable initial conditions.

6. PAUSING AT A SINGULARITY

Time indeterminacy also occurs at singularities encountered as a solution evolves along a discontinuity surface. Although some cases have been noted (e.g. [7]), as in continuous systems their time ambiguities have not been resolved previously to the authors' knowledge. We present here a fundamental example that has arisen not only in classifications of bifurcations (e.g. in [7]), but in applications such as genetics

(e.g. [11]). We then show two other examples, induced by a discontinuity or a singularity in the vector field, that turn out to be just the classic example from section 3 in disguise.

6.1. Fold-fold singularity, a second example from genetic regulation. In studying translation-transcription models in the regulation of genetic networks, systems occur that are similar to section 4.2 but include an additional mRNA stage. A class of such models were observed to exhibit an indeterminacy in [11], which we will show is another example of arbitrary pausing. The prototype of a one-gene model, describing the concentrations of mRNA and protein products by x and y respectively, is given as

$$\left. \begin{array}{l} \dot{x} = z - x \\ \dot{y} = 3x - y \end{array} \right\} \quad z = H(y - 2). \quad (43)$$

The step function z represents the switching on/off of production of the mRNA when the protein concentration is above/below a threshold $y = 2$ (the switch being the limit of a Hill function as in section 4.2). The observation made in [11] is essentially that the family of functions

$$\phi_{\tau}^{\zeta}(t) = \begin{cases} (2e^{-t}/3, 2e^{-t}(t+1)) & \text{if } t < 0, \\ (2/3, 2) & \text{if } 0 \leq t \leq \tau, \\ \left(\begin{array}{l} \zeta + (\frac{2}{3} - \zeta)e^{-(t-\tau)}, \\ 3\zeta + (2 - 3\zeta)e^{-(t-\tau)}(1 + t - \tau) \end{array} \right) & \text{if } t > \tau, \end{cases} \quad (44)$$

with $\tau > 0$, are solutions $x(t) = \phi_{\tau}^{\zeta}(t)$ for any τ , with two choices of exit trajectory $\zeta = 0$ or $\zeta = 1$. As pictured in fig. 5, the trajectory begins in the lower-right region of state space, where $x > 2/3$ and $y < 2$. (There is a solution similar to (44) starting in the upper-left region.)

The special point (x, y) is a point where $\dot{y} = 0$ on the surface $y = 2$ for both the upper ($z = 1$) and lower ($z = 0$) systems, meaning the vector field is tangent to the discontinuity surface $y = 2$ from both above and below.

To resolve both the spatial indeterminacy from the choice of exit direction via $\zeta = 0$ or 1 , and the temporal indeterminacy in the exit time τ , we examine the blow-up of the discontinuity. To do this we map the interval $y \in [2 - \varepsilon, 2 + \varepsilon] \rightarrow 2$ for $\varepsilon \rightarrow 0$ onto the interval $z \in [0, 1]$, across the discontinuity in (43), by letting $y - 2 = \varepsilon z$ on $z \in [0, 1]$ for $\varepsilon \rightarrow 0$. Substituting in to (44), the transition of z between 0 and 1 on $y = 2$ is then given by $\varepsilon \dot{z} = 3x - y$. Together with the \dot{x}

equation, on $y = 2$ we have

$$\left. \begin{aligned} \dot{x} &= z - x \\ \varepsilon \dot{z} &= 3x - 2 - 2\varepsilon z \end{aligned} \right\} \quad z \in (0, 1). \quad (45)$$

The $O(\varepsilon)$ term on the righthand side constitutes a regular perturbation that will vanish as we let $\varepsilon \rightarrow 0$, so for brevity let us omit it from our calculations (unlike the ε on the lefthand side which is a singular perturbation, and so is not trivial for $\varepsilon \rightarrow 0$). This system has a saddle equilibrium at $x = z = 2/3$, through which the stable manifold has gradient $z/x = \frac{1}{2} - \rho$ where $\rho = \frac{1}{2}\sqrt{1 + 12\varepsilon^{-1}}$. This touches the boundary of the blow-up interval, $z = 0$, at $x = x_c := 2(\rho + \frac{1}{2})/3(\rho - \frac{1}{2})$. As illustrated in fig. 5, solutions will evolve via the $\zeta = 0$ or $\zeta = 1$ solution through the discontinuity depending which side of the stable manifold they hit the discontinuity.

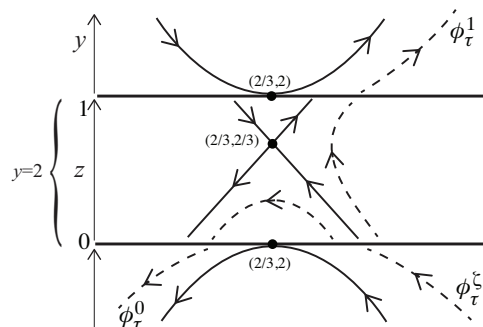


FIGURE 5. The genetic fold-fold system ('fold' refers to the turning around of the flows in $y > 0$ and $y < 0$), and its blow-up interval for z , showing a trajectory ϕ_τ^ζ which splits after the discontinuity into ϕ_τ^1 or ϕ_τ^0 , depending which side of the saddle's stable manifold it lands. If ϕ_τ^ζ hits the discontinuity close to the stable manifold then it takes an arbitrarily long time τ to cross the discontinuity, away from the manifold the transition time is $\tau = 0$.

Solutions to (45) that begin on $z = 0$ at $x = x_0$ at time $t = 0$ are given by

$$\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = c_+ e^{(-\frac{1}{2}-\rho)t} \begin{pmatrix} 1 \\ \frac{1}{2} - \rho \end{pmatrix} + c_- e^{(-\frac{1}{2}+\rho)t} \begin{pmatrix} 1 \\ \frac{1}{2} + \rho \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}. \quad (46)$$

where $c_\pm = \frac{6\rho x_0 - 4\rho \pm (2 + 3x_0)}{12\rho}$. The time $t = \tau$ at which the $z(t)$ solution returns to $z = 0$ or $z = 1$ is then given by

$$\{0 \text{ or } 1\} = c_+ e^{(-\frac{1}{2}-\rho)\tau} (\frac{1}{2} - \rho) + c_- e^{(-\frac{1}{2}+\rho)\tau} (\frac{1}{2} + \rho) + \frac{2}{3}.$$

The $e^{-\rho\tau}$ term above is small since $\rho = \mathcal{O}(\varepsilon^{-1/2})$ and $\tau > 0$, so we can approximate it. We consider initial points $x_0 = x_c \pm e^{-B(-\frac{1}{2}+\rho)}$ for some constant B , with the $+$ or $-$ signs corresponding to end conditions $\{0$ or $1\}$, then after substituting in for x_0 , a little algebra gives

$$\begin{aligned} \tau &= B + \frac{1}{\rho - \frac{1}{2}} \log \left[\frac{2 \{- \text{ or } +\} \rho}{\rho^2 - \frac{1}{4}} (\{0 \text{ or } 1\} - \frac{2}{3}) - \mathcal{O}(\rho e^{-\rho B}) \right] \\ &= B + \sqrt{\frac{\varepsilon}{3}} \log \left[2 \left\{ \frac{2}{3} \text{ or } \frac{1}{3} \right\} \sqrt{\frac{\varepsilon}{3}} \right] + \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}} e^{-B/\sqrt{\varepsilon}}\right) \\ &= B + \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}} e^{-B/\sqrt{\varepsilon}}, \sqrt{\varepsilon} \log \varepsilon\right). \end{aligned} \quad (47)$$

Thus if the solution hits the discontinuity at $x = x_c$, then by approaching this limit with x exponentially close to x_c as $\varepsilon \rightarrow 0$, we find that the time to pass through the point $(x, y) = (2/3, 2)$ is an arbitrary value $\tau = B$.

6.2. Fold-fold singularity. A vector field that not only has a discontinuity, but is also non-analytic elsewhere, may exhibit time indeterminacy in its sliding dynamics. Consider the system

$$(\dot{x}, \dot{y}) = \frac{1}{2}(1+u)(1, x) + \frac{1}{2}(1-u)(2|x|^\gamma - 1, -x), \quad (48)$$

with $u = \text{sign}(y)$. This has a singularity at $x = y = 0$, similar to the genetic example above, called a *fold-fold* singularity, where $y = \dot{y} = 0$ and $\ddot{y} \neq 0$ for both the upper vector field $(1, x)$ (with $u = +1$) and the lower vector field $(2|x|^\gamma - 1, -x)$ (with $u = -1$).

The blow-up dynamics at $y = 0$, obtained by letting $y = \varepsilon u$ on $u \in [-1, +1]$, is $\varepsilon \dot{u} = \frac{1}{2}(1+u)x + \frac{1}{2}(1-u)(-x) = ux$ for a non-negative infinitesimal ε . This has solutions that ‘slide’ along $u = 0$, and therefore satisfy $\dot{u} = 0$. The dynamics on $y = 0$ is then $\dot{x} = |x|^\gamma$, which is just (7), and therefore exhibits the time indeterminacy described in section 3, with arbitrary pausing observed at $\gamma = 1$ in the absence of hidden terms, and at any value of γ if the presence of hidden terms induces bifurcations inside the blow-up dynamics at $x = 0$.

6.3. Singularity in the vector field. Our final example shows indeterminacy arising from an infinity in the vector field, and leading to an alternative extension of the classic example (7).

Consider the system

$$\dot{y} = -y^2/z^2, \quad \dot{z} = -2y/3z. \quad (49)$$

Although this is infinite all along $z = 0$, let us concern ourselves only with initial conditions in $z \neq 0$, from which solutions can intersect the

line $z = 0$ only at the origin, $y = z = 0$. Those solutions lie on curves which satisfy

$$\frac{dz}{dy} = \frac{\dot{z}}{\dot{y}} = \frac{2z}{3y} \quad \Rightarrow \quad Cz = |y|^{2/3}, \quad (50)$$

for any constant C . The resulting phase portrait is shown in fig. 6. Solutions to (49) therefore exist and are unique away from the line $z = 0$. The set $y = 0$ is a degenerate line of stationary points.

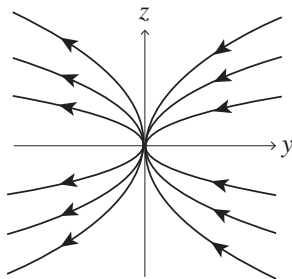


FIGURE 6. Non-unique trajectories of (49). The phase portrait suggests that solutions entering the origin from the right half plane, can depart along any trajectory into the left half plane, possibly with an arbitrarily long pause in between.

By differentiating the last equation in (50) we find that the evolution of y and z restricted to the curves (50) is given by

$$\begin{aligned} \dot{y} &= -|C|^2 |y|^{2/3}, \\ \dot{z} &= -\frac{2}{3} |C|^{3/2} |z|^{1/2} \text{sign}(yz). \end{aligned} \quad (51)$$

The \dot{y} and \dot{z} equations are decoupled, and are each equivalent to the classic example (7) for different powers γ . Solutions of (51) will therefore consist of functions of the form (8) (up to a time rescaling), and result in pausing similar to section 3.

The key difference to the classic example is the added spatial dimension. One way to handle this is to introduce new coordinates (Y, Z) for which

$$y = r^2 Y, \quad z = rZ, \quad r \rightarrow 0, \quad (52)$$

which blows up the point $(y, z) = (0, 0)$ into $Y^2 + Z^2 = y^2/r^4 + z^2/r^2 = 1$, a vanishingly small ellipse in the real space of (y, z) , and a non-vanishing circle in (Y, Z) space. In the (Y, Z) coordinates one may then ascertain the dynamics through the point $(y, z) = (0, 0)$, and show that arbitrary pausing occurs.

7. CONCLUSIONS

We have described a number of examples of indeterminacy, the common methods used to resolve them, and their interpretation in terms of arbitrary pauses in motion.

In systems with a discontinuity associated with a fractional power law (i.e. $|x|^\alpha$), or a singularity (i.e. $1/x^k$), blowing up the critical point reveals an equilibrium for certain values of the exponent (α or k). Time indeterminacy in the solution can be resolved, except at special values of the exponent where the equilibrium undergoes a bifurcation. Spatial indeterminacy can also be at least partially resolved by the blow-up.

Indeterminacy at a discontinuity typically involves a saddle equilibrium, which permits both spatial and time indeterminacy to be resolved, except for an exponentially small range of parameters for which solutions approach the saddle's stable manifold, where pausing for an arbitrary time occurs.

We hope these investigations provide a window into pausing as a general phenomenon, and suggest preliminary steps towards more probing models of such behaviour in physical and biological processes. While it seems likely that such re-scalings have been performed before to study indeterminacy in the classic cases like (7), we are not aware of a discussion of a general method, or of the conditions that create pausing, particularly in a range of situations that include indeterminacies arising from discontinuity in the vector field, in its derivative, or from singular points in a vector field.

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