ON THE DULAC'S PROBLEM FOR PIECEWISE ANALYTIC VECTOR FIELDS

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Abstract. This paper attempts to present a version of Dulac's problem for piecewise analytic vector fields that states conditions for the birth of the number of limit cycles around certain minimal sets. A suitable model-theoretic structure is introduced under which a qualitative investigation of the problem is settled.

1. Introduction

In short, recall that Dulac's Problem consists in studying if an elementary polycycle of an analytic vector field admits limit cycles accumulationg onto it. This problem raises from an attempt of studying how many limit cycles can be admitted by a polynomial vector field of degree n on $\mathbb{R}^2$, i.e., the second part of the Hilbert's sixteenth problem. From the original work of Dulac it is derived that if a polynomial differential equation has saddle connections forming a simple homoclinic or heteroclinic loop, then the equation has finitely many limit cycles. More specifically, the second part of the Hilbert's sixteenth problem can be written as:

- *Determine the maximum number of limit cycles admitted by a polynomial vector field of degree n on $\mathbb{R}^2$.*

A preliminary step towards the solution of this part of Hilbert's sixteenth problem is proving the following result:

- *A polynomial vector field on $\mathbb{R}^2$ has at most a finite number of limit cycles.*

This last question can be extended to analytic vector fields and it can be reduced to the problem of non-accumulation of limit cycles, see [19]:

- *An elementary polycycle of an analytic vector field cannot have limit cycles accumulating onto it.*

In 1923, the French mathematician Henri Dulac gave an incomplete proof which was noticed much later, thus it turned out to be called the Dulac's Problem. A correct proof was given for quadratic vector fields by R. Bamon, [2], and the general case was estated independently by Yu Il'Yashenko, [12], and by J. Ecalle, [8].

In addition to the study in smooth systems, cycles can be studied in the context of piecewise smooth vector fields. In the last years, the theory of discontinuous vector fields has become stronger with growing importance at the frontier between mathematics, physics, engineering, and the life sciences. Interest stems, particularly, from piecewise smooth dynamical models in control theory [3], nonlinear oscillations [1, 16], impact and friction mechanics [6], economics [14, 11], biology [4], etc. In the present work, we consider bifurcations of a planar vector field Z presenting discontinuities on a line L. Such system can have loops represented by typical singularities connections where such singularities are in L resulting in scenarios that do not appear for smooth vector fields.

Let U be an open subset of $\mathbb{R}^2$ with compact closure. Consider a smooth embedded submanifold $\Sigma = h^{-1}(0) \cap U$, where $h : U \to \mathbb{R}$ is a smooth function for which 0 is a regular value. In this way, Σ splits U in two open regions

$$\Sigma^+ = \{p \in U; h(p) > 0\} \qquad \text{and} \qquad \Sigma^- = \{p \in U; h(p) < 0\}.$$

A piecewise analytic vector field in U is as vector field of the form

$$Z(p) = \begin{cases} X(p), & p \in \Sigma^+, \\ Y(p), & p \in \Sigma^-, \end{cases}$$

where X and Y are analytic vector fields in U. Denote by Ω^ω the set of all piecewise analytic vector fields defined as above. In Σ the following regions are distinguished

- crossing region: $\Sigma^c = \{p \in \Sigma;\ Xh \cdot Yh(p) > 0\}$,
- sliding region: $\Sigma^s = \{p \in \Sigma;\ Xh(p) < 0 \text{ and } Yh(p) > 0\}$,
- escaping region: $\Sigma^e = \{p \in \Sigma;\ Xh(p) > 0 \text{ and } Yh(p) < 0\}$,

where $Xh(p) = \langle X, \nabla h\rangle(p)$ is the Lie derivative of h, at p, in the direction of X, the same for $Yh(p)$. Trajectories of Z through points on the switching manifold Σ follow the Filippov convention, see [9, 10]. It means that, the trajectory of Z through $p \in \Sigma^c$ is the concatenation of trajectories of X and Y through p. For $p \in \Sigma^s \cup \Sigma^e$ the trajectory obeys the Filippov vector field obtained by the convex combination of $X(p)$ and $Y(p)$ which is tangent to Σ at p.

As seen above, Dulac's problem was originally proposed for analytic vector fields. In this context, polycycles admit hyperbolic and elementary singularities. A next step in this direction is to replace analytic by piecewise analytic vector fields. So, we propose a version of this problem for piecewise analytic vector fields considering polycycles possessing only hyperbolic singularities and crossing Σ only in points of $\overline{\Sigma^c}$.

Consider a piecewise analytic vector field $Z = (X, Y)$ where X and Y are analytic vector fields in $\mathbb{R}^2$ and assume that Z admits a hyperbolic polycycle Γ, i.e., Γ is a closed curve composed by a finite number of segments of regular orbits and a finite number of hyperbolic saddles of $Z = (X, Y)$, i.e., hyperbolic saddles of X in $\overline{\Sigma^+}$ or hyperbolic saddles of Y in $\overline{\Sigma^-}$, see Definition 1. If $p \in \Sigma$ is a hyperbolic saddle point of X (resp. Y) and a regular point o Y (resp. X) then p is said to be a saddle-regular point of Z. If $p \in \Sigma$ is a hyperbolic saddle of both X and Y, then p is called a saddle-saddle point of Z.

The question we want to answer is

- *Can a hyperbolic polycycle Γ of Z have limit cycles accumulating onto it?*

The answer for this question is no provided the polycycle is hyperbolic. If a polycycle Γ does not intersect the switching manifold, then the problem reduces to the smooth one. Thus, we also suppose $\Gamma \cap \Sigma \neq \emptyset$. In order to answer this question we follow the same steps for the smooth case in [19]. Our objective is to give an extension for piecewise analytic vector fields of all concepts and results existent for hyperbolic polycycles of analytic vector fields. It is worthwhile to emphasize the extension performed is not a particular case of the original problem, there exist essential differences in the transition maps near hyperbolic saddle points. Moreover, we generalize the result by allowing fold points on the polycycle, i.e., by allowing that some of the singularities of the polycycle to be points where the contact between the vector field and the switching manifold Σ is of order 2.

In another words, the central point here is to prove the following theorems.

Theorem A. *A hyperbolic polycycle of a piecewise analytic vector field $Z \in \Omega^\omega$ cannot have limit cycles accumulating onto it.*

Theorem B. *A polycycle of a piecewise analytic vector field $Z = (X, Y) \in \Omega^\omega$, which singularities are only hyperbolic saddles outside of the switching manifold, saddle-regular, saddle-saddle, fold-regular, fold-fold, and fold-saddle points, cannot have limit cycles accumulating onto it.*

This work is organized as following. Section 2 is devoted to state notations and construction of the first return map to be studied. Theorem A is proved in Section 3. Finally, in Section 4 the result presented in Theorem A is generalized for polycycles presenting fold points on the switching manifold and Theorem B is proved.

2. Preliminaries

Firstly, it is necessary to state a good normal form for a vector field near a hyperbolic saddle point. Consider the vector field $X \in \chi^\infty$ and suppose that X has a hyperbolic saddle $s \in \mathbb{R}^2$. The main interest here is to study X in a neighborhood of s so, without loss of generality, we suppose $s = (0, 0)$ and X is defined in a neighborhood of s, $\mathcal{V}_0 \subset \mathbb{R}^2$. Due to the hyperbolicity we also assume s is the unique singular point of X in $\mathcal{V}_0$.

Let μ_1 and μ_2 be the eigenvalues of $DX(s)$ with $\mu_2 < 0 < \mu_1$ and let $r = -\dfrac{\mu_2}{\mu_1}$ be the ratio of hyperbolicity of X at s. The next theorem is explored in [19] by combining results due to Bonckaert and the Poincaré-Dulac normal form, see [5, 18, 19].

Theorem 1. *Let X be a C^∞ having a hyperbolic saddle point at the origin with hyperbolicity ratio r. Then, there exists a function $N : \mathbb{N} \to \mathbb{N}$ such that, in some neighborhood of the saddle point s, X is C^k-conjugated to the polynomial vector field*

$$x\frac{\partial}{\partial x} + \left(-r + \frac{1}{q}\sum_{i=1}^{N(k)} \alpha_{i+1}(x^p y^q)^i\right) y\frac{\partial}{\partial y}, \tag{1}$$

if $r = \frac{p}{q} \in \mathbb{Q}$. If $r \notin \mathbb{Q}$, X is C^k-conjugated to the linear vector field

$$x\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}. \tag{2}$$

Remark 1. *More generally, if X is an analytic vector field and $r = p/q \in \mathbb{Q}$, then X is conjugated to the following analytic vector field*

$$x\frac{\partial}{\partial x} + \frac{1}{q}\left(-p + \sum_{i=0}^{\infty} \alpha_{i+1}(x^p y^q)^i\right) y\frac{\partial}{\partial y}, \tag{3}$$

where $P_\alpha(z) = \sum_{i=1}^{\infty} \alpha_i z^i$ is an analytic function of $z \in \mathbb{R}$, and $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{A}$, where $\mathcal{A}$ is the set

$$\mathcal{A} = \left\{\alpha = (\alpha_1, \alpha_2, \ldots);\ |\alpha_1| < \frac{1}{2}, |\alpha_i| < M \text{ for } i \geq 2\right\}$$

and $M > 0$ is a fixed constant, see [19].

Now, remember that Ω^ω denote the set of all piecewise analytic vector fields in $\mathbb{R}^2$, i.e., for all $Z = (X, Y) \in \Omega^\omega$ the vector fields X and Y are analytics.

Definition 1. *Let $Z = (X, Y) \in \Omega^\omega$ be a piecewise analytic vector field. A continuous closed curve Γ is said to be a hyperbolic polycycle of the vector field Z if it is composed by a finite union of segments of regular orbits, $\gamma_1, \gamma_2, \ldots, \gamma_n$, of Z and hyperbolic saddles, $p_1, p_2, \ldots, p_n$, such that for each $1 \leq i \leq n$, the ending points of γ_i are p_i and p_{i+1}, respectively. Moreover, $\gamma_i \cap \Sigma \subset \Sigma^c$, for all $i = 1, \ldots, n$, if $p_i \in \Sigma$ then $p_i \in \overline{\Sigma^c}$ and p_i is a saddle-regular or saddle-saddle point of Z for which the invariant manifolds of the saddle are transversal to Σ at this point.*

Under these conditions, there exists a first return map defined in one of the regions delimited by the polycycle. Before the analysis of the first return map some technical concepts are required. The following definitions concern about real maps defined in a half-open interval.

Definition 2. *The Dulac series of a map f, defined in a half open interval $[0, \delta)$, is a formal series*

$$\widehat{f}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(\ln x),$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ is an increasing sequence of positive numbers tending to infinity, P_i are polynomials. Moreover, this series must be asymptotic to f in the following sense: for any $n \in \mathbb{N}$, there is $s \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{i=1}^{s} x^{\lambda_i} P_i(\ln x) \right| = O(x^{\lambda_n}).$$

Definition 3. *A germ of a map f at $0 \in \mathbb{R}^+$ is called quasi-regular if*

1. *f has a representative on $[0, \delta)$ which is C^∞ on $(0, \delta)$, where δ is a positive constant;*
2. *f admits a Dulac series.*

Moreover, f is a quasi-regular homeomorphism if it is quasi-regular and $P_1 \equiv A$, where A is a positive constant.

It follows from the definition that the set $\mathcal{D}$ of the quasi-regular homeomorphisms is a group (with the composition of maps) which contains the group Diff_0 of germs of diffeomorphisms fixing the origin, see [19].

Definition 4. *A germ of a map $f : [0, \delta) \to \mathbb{R}$ is quasi-analytic if*

1. *f is quasi-regular;*
2. *the map $x \to f \circ \exp(-x)$ has a bounded holomorphic extension $F(z)$ in some domain Δ_b of $\mathbb{C}$, defined by $\Delta_b = \{z = u + iv \in \mathbb{C};\, u > b(1 + v^2)^{1/4}\}$ where b is a positive real number.*

The first step towards the solution of the proposed problem is to construct a first return map. Let Γ be a hyperbolic polycycle of $Z \in \Omega^\omega$ and let $p_1, \ldots, p_n$ be the vertices of Γ, in some cyclic order. Let $\sigma' \simeq [a, b)$ be a half open segment transversal to Γ, with $a \in \Gamma$, and such that the first return map P of Γ is defined from $\sigma \subset \sigma' \to \sigma'$.

At each vertex p_i choose a local system of coordinates (x_i, y_i) in such way that the axis 0_{x_i} and 0_{y_i} are the local unstable and stable manifolds, respectively. Moreover,

if $p_i \in \Sigma$ we assume that Σ is a curve crossing transversely both axes at the origin and the first return map P is defined in the first quadrant of the $(x_i, y_i)-$plane. Denote by Σ_i^+ the half-plane that contains the positive part of axis 0_{y_i} and by Σ_i^- the half-plane that contains the negative part of axis 0_{y_i}. Consider transversal sections σ_i and τ_i defined as following:

Case C_1- $p_i \notin \Sigma$ or $p_i \in \Sigma$ and Σ does not lie on the first quadrant: let σ_i and τ_i be transversal sections to 0_{y_i} at $(0, \varepsilon_i)$ and to 0_{x_i} at $(\varepsilon_i, 0)$, respectively with $\varepsilon_i > 0$. See Figure 1 (a) and (b);

Case C_2- $p_i \in \Sigma$, Σ lies on the first quadrant and p_i is a saddle of the vector field defined in Σ_i^+: let σ_i be a transversal section to 0_{y_i} at $(0, \varepsilon_i)$ with $\varepsilon_i > 0$ and let τ_i be a segment contained in Σ with $0 \in \tau_i$. See Figure 1 (c);

Case C_3- $p_i \in \Sigma$, Σ lies on the first quadrant and p_i is a saddle of the vector field defined in Σ_i^-: let τ_i be a transversal section to 0_{x_i} at $(\varepsilon_i, 0)$ with $\varepsilon_i > 0$ and let σ_i be a segment contained in Σ with $0 \in \sigma_i$. See Figure 1 (d).

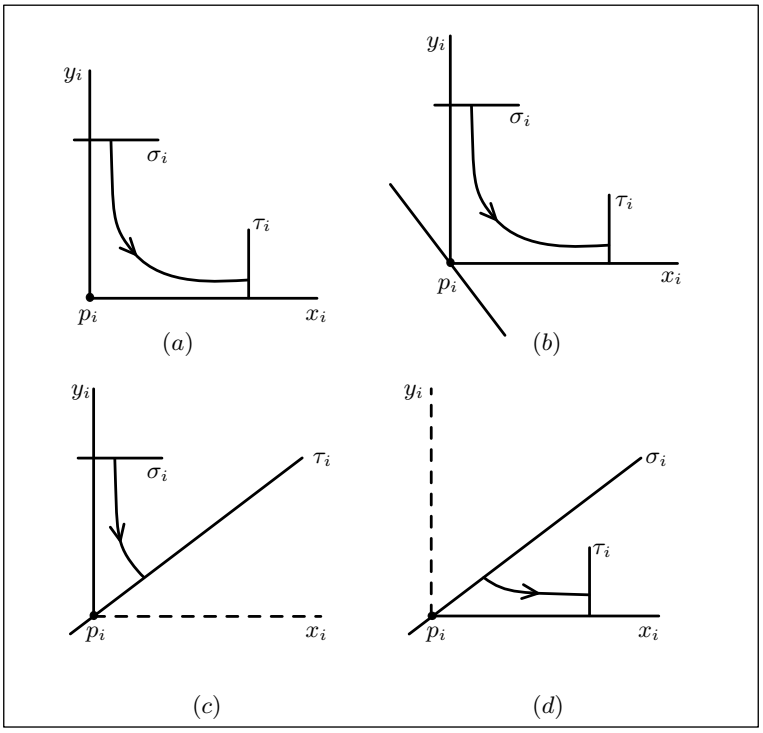

FIGURE 1. Transition maps from σ_i to τ_i: $(a) - (b)$ Case 1, (c) Case 2 and (d) Case 3.

In each one of these cases, one define a transition map D_{ij}, $j = 1, 2, 3$, near the saddle point p_i, from σ_i^+ to τ_i, where $\sigma_i^+ \subset \sigma_i$ with $x_i \geq 0$ and $D_i(0) = 0$. Since quasi-analyticity is preserved by diffeomorphism, there is no loss of generality in particularizing transversal sections and in assuming $\varepsilon_i = 1$, for all i. In the cases C_1 and C_2, consider σ_i as the image of the map $x_i \in (-\delta_i, \delta_i) \mapsto (x, 1) \in \mathbb{R}^2$, where $\delta_i > 0$ is small enough. In the case C_3 we consider σ_i given by the parametrization of Σ. Let $\varphi_i(t, x_i, y_i)$ be the flow near the saddle and let $t_i(x_i)$ be the smallest positive time spent by the trajectory passing through $(x_i, y_i) \in \sigma_i^+$ to intersect τ_i, $t_i(x_i)$

is called transition time from σ_i to τ_i. Then, define $D_{i2}(x) = \pi_1(\varphi_i(t_i(x_i), x_i, 1))$ and $D_{ij}(x) = \pi_2(\varphi_i(t_i(x_i), x_i, \varepsilon_i))$ for $j = 1, 3$ where π_1 and π_2 are the canonical projections on the first and second coordinate, respectively.

After crossing τ_i, the trajectory of Z by $p \in \sigma_i$ will cross σ_{i+1}. If $\tau_i \cap \sigma_{i+1} = \emptyset$, since a regular orbit makes the connection between p_i and p_{i+1}, there exists an analytic diffeomorphism $R_i : \tau_i \to \sigma_{i+1}$ such that $R_i(q)$ is the point in σ_{i+1} where the positive trajectory of Z passing through $q \in \tau_i$ intersects σ_{i+1}. If $\tau_i \cap \sigma_{i+1} \neq \emptyset$, the transition from τ_i to σ_{i+1} will be just the identity map and for questions of completeness we also denote it by R_i. See Figure 2 as an illustration of this construction. Finally, the first return map can be written as a composition

$$P(x) = R_n \circ D_{nj_n} \circ \cdots \circ R_1 \circ D_{1j_1}(x), \tag{4}$$

with $j_k = 1$, 2 or 3 for $k = 1, \ldots, n$.

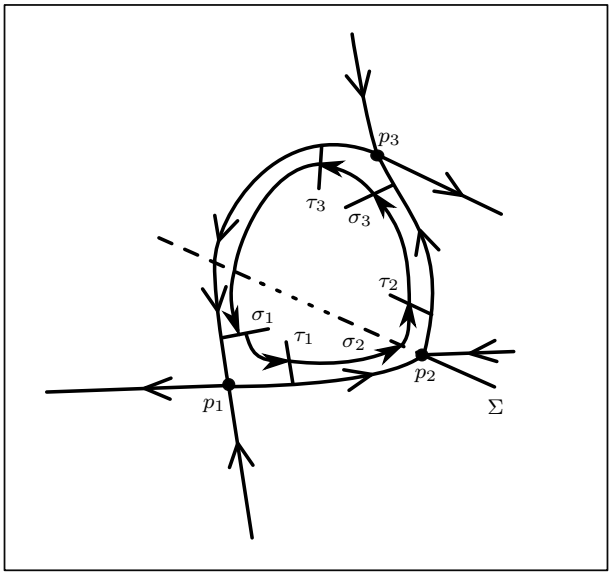

FIGURE 2. First return map for a polycycle with 3 saddle points. Here $\sigma_2 \subset \Sigma$.

Remark 2. *Observe that the structure of the first return map for polycyles of piecewise analytic vector field is very similar to that for analytic vector fields. The central difference is the existence of two different types transition maps, D_{i2} and D_{i3}, that does not happen in the analytic case.*

Remark 3. *In order to simplify the notation, the index i will be omitted in D_{ij}, i.e., $D_{ij} = D_j$. It is possible once we analysis is local at a fixed saddle point.*

3. Proof of Theorem A

A sufficient condition to prove that a hyperbolic polycycle of an analytic vector field cannot have limit cycles accumulating onto it is to prove that the first return map near a hyperbolic polycycle is quasi-analytic, see Chapter 3 in [19]. As seen in Remark 2, the structure of the first return map in the piecewise analytic case is the same as in the analytic case, a sufficient condition to prove Theorem A is to prove

that the first return map remains quasi-analytic. Moreover, as seen in Chapter 3 of [19], in order to prove that the first return map is quasi-analytic it is sufficient to prove that the transition maps at hyperbolic saddle points are quasi-analytic. In fact, the piecewise analytic case differs from the analytic one by the existence of transition maps of the types D_2 and D_3. Therefore, the only thing that must be proved is that transition maps as D_2 and D_3 are quasi-analytic maps. In order to prove it, we follow the same steps used in [19] to prove that D_1 is a quasi-analytic map, for this reason, we omit some details.

The first condition in the quasi-analyticity definition is quasi-regularity. So, now we prove that the transition maps at hyperbolic saddle points are quasi-regular maps.

Proposition 1. *Transition maps associated to hyperbolic saddle points of piecewise analytic vector fields are quasi-regular homeomorphisms.*

Proof. Let $Z = (X, Y)$ be a piecewise analytic vector field. Without loss of generality suppose that X has a hyperbolic saddle point at the origin. So, we want to study the transition map associated with this saddle point. This transition map can be of type D_i, $i = 1, 2, 3$.

If the transition map is of type D_1 then it coincides with the classical case which is already proved, see for instance [13, 19].

Now, observe that $D_1 = D_3 \circ D_2$, i.e., a map of type D_1 is a composition of maps of type D_2 and D_3. Since $\mathcal{D}$ is a group and $D_1 \in \mathcal{D}$ it is enough to prove one of the statements: $D_2 \in \mathcal{D}$ or $D_3 \in \mathcal{D}$.

In order to prove that $D_3 \in \mathcal{D}$ suppose that the saddle point of X is at the origin of the system of coordinates (x, y) and consider V_0 be the neighborhood of the saddle point where the normal form given in equation (3) holds. Since the analysis is local we can suppose that Σ is locally a straight line given by $\Sigma = \{(x, y) \in \mathbb{R}^2; y = x\}$. Furthermore, as seen before, we can consider $\sigma_3 = \{(x, x) \in \mathbb{R}^2; x \in (-\delta, \delta)\} \subset \Sigma$ and $\tau_3 = \{(1, y) \in \mathbb{R}^2; |y| < \bar{\delta}\} \subset \Sigma$, for some $\delta > 0$, $\bar{\delta} > 0$. Denote by $\varphi(t, x, y) = (\varphi_1(t, x, y), \varphi_2(t, x, y))$ the flow of the vector field X defined in Σ^-, then $\varphi_1(t, x, y) = e^t x$. So, the transition time from σ_3 to τ_3 is $t_3(x) = -\ln x$ and $D_3(x) = \varphi_2(t_3(x), x, x)$.

The normal form considered has two different expressions provided $r \in \mathbb{Q}$ or $r \notin \mathbb{Q}$. For this reason we have two different situations to explore:

I) If $r \notin \mathbb{Q}$: in this case, $\varphi_2(t, x, y) = e^{-rt} y$. Therefore, $D_3(x) = e^{-rt_3(x)} x = x^{1+r}$ which is an analytic map. Consider the following Dulac series

$$\widehat{D}_3(x) = \sum_{i=1}^{\infty} x^{\lambda_i^3} P_i^3(\ln x),$$

where $\lambda_1^3 < \lambda_2^3 < \cdots$ is an increasing sequence tending to infinity with $\lambda_1^3 = 1 + r$, $P_1^3(u) \equiv 1$ and $P_i^3(u) \equiv 0$ for all $i > 1$. Hence, $D_3(x)$ is a quasi-regular homeomorphism.

Hence, D_3 is a quasi-regular homeomorphism.

II) If $r = \dfrac{p}{q} \in \mathbb{Q}$, with p and q without common factors: it is not possible to obtain algebraically a expression for $D_3(x)$. In this case, by performing the singular change of coordinates $x = x$, $u = x^p y^q$, the differential equation associated with X

becomes

$$\begin{cases} \dot{x} = & x \\ \dot{u} = & P_\alpha(u) = \sum_{i=1}^{\infty} \alpha_i u^i. \end{cases} \tag{5}$$

Now, system (5) has separable variables and $\varphi_1(t,x) = e^t x$ is the solution for the first equation satisfying $\varphi_1(0,x) = x$. For the second one, P_α is analytic for $\alpha \in \mathcal{A}$. Let $\varphi_2(t,u)$ be the solution of this equation, which is analytic, verifying $\varphi_2(0,u) = u$.

We can expand $\varphi_2(t,u)$ in series in u for each t,

$$\varphi_2(t,u) = \sum_{i=1}^{\infty} g_i(t) u^i. \tag{6}$$

We know that, the time spent to go from σ_i to τ_i is $t_i(x) = -\ln x$, $i = 1, 3$. Therefore, we have $u_{|\sigma_3} = x^p x^q = x^{p+q}$ and $u_{|\tau_3} = (e^{t(x)}x)^p D_3(x)^q = D_3(x)^q$. Also, $u_{|\tau_3} = u(t(x), u_{|\sigma_3})$, i.e.,

$$D_3(x)^q \quad = u_{|\tau_3} = u\left(-\ln x, x^{p+q}\right) = \sum_{i\geq 1} g_i\left(-\ln x\right) x^{(p+q)i}, \tag{7}$$

for $x > 0$, provided this series converges for $t = -\ln x$, and $D_3(0) = 0$.

By performing a similar analysis for a transition maps of type D_1 we obtain

$$D_1(x)^q \quad = u_{|\tau_1} = u\left(-\ln x, x^p\right) = \sum_{i\geq 1} g_i\left(-\ln x\right) x^{pi}, \tag{8}$$

for $x > 0$, provided this series converges for $t = -\ln x$, and $D_1(0) = 0$. Hence, D_1^q and D_3^q has the same structure, it implies that the proof of D_3 is a quasi-regular map follows exactly in the same way as the proof of D_1 is quasi-regular, see Chapter 5 in [19] and Chapter 2 in [7]. □

Finally, to finish the proof of Theorem 1 it is enough to prove the following result.

Proposition 2. *The transition maps associated to hyperbolic saddle points of piecewise analytic vector fields, D_i, $i = 1, 2, 3$, are quasi-analytic maps.*

Proof. We consider the same statements as in proof of Proposition 1 and it follows from that result that it is enough to prove the second part of Definition 4. It means that we have to prove that $f_i(x) = D_i(e^{-x})$, has a bounded holomorphic extension F_i in some domain $\Delta_{b_i} = \{x + iy \in \mathbb{C};\, x > b_i(1 + y^2)^{1/4}\}$, with $b_i > 0$ constant, for $i = 1, 2, 3$. Similarly to the proof of Proposition 1, the case $i = 1$ coincides with the classical one, see [19].

Since D_1 is quasi-analytic, the map f_1 has a bounded holomorphic extension F_1 in $\Delta_{b_1} = \{x + iy \in \mathbb{C};\, x > b(1 + y^2)^{1/4}\}$ for some $b_1 > 0$. Now, we prove that D_3 is quasi-analytic, in order to do that consider the maps

$$E : x \in \mathbb{R} \to e^{-x} \quad \text{and} \quad L : x \in \mathbb{R}_+^* \to -\ln x,$$

where $\mathbb{R}_+^* = \{u \in \mathbb{R}; u > 0\}$. Observe that $L = E^{-1}$. From normal form 9 we obtain $D_2(x) = e^{-t_2(x)}x$ where $t_2(x)$ is the transition time from $\sigma_2^+ = \{(x, 1) \in \mathbb{R}^2; x > 0\}$ to $\tau_2 = \{(x, y) \in \mathbb{R}^2; y = x\}$. It is known that $D_3 \circ D_2 = D_1$ and D_i are invertible maps, for $i = 1, 2, 3$, then

$$f_3 = D_3 \circ E = (D_1 \circ E) \circ (L \circ D_2^{-1} \circ E) = f_1 \circ h,$$

where $h = L \circ D_2^{-1} \circ E$. For $x + iy \in \mathbb{C}$ define $H(x + iy) = h(x) + iy$ and $F_3(x+iy) = F_1 \circ H(x+iy)$. Notice that $F_3|_{\mathbb{R}} = f_3$, so F_3 is a complex extension of f_3. Moreover, since h is a smooth map for $x > 0$, it follows that H is holomorphic in $\{x + iy \in \mathbb{C}; x > 0\}$.

Now, we show that F_3 is bounded in Δ_{b_1}. In order to do that, observe that $D_2^{-1}(u) \leq u$ for all $0 < u \leq 1$, L is a decreasing map and $0 < E(x) \leq 1$ for all $x > 0$. It implies that $h(x) \geq x$ for all $x > 0$. Consider $x + iy \in \Delta_{b_1}$, then $h(x) \geq x > b_1(1 + y^2)^{1/4} \geq 0$ and, consequently, $H(x + iy) = h(x) + iy \in \Delta_{b_1}$ and $H(\Delta_{b_1}) \subset \Delta_{b_1}$. Since F_1 is bounded in Δ_{b_1} and $F_3 = F_1 \circ H$, it follows that F_3 is bounded and holomorphic in Δ_{b_1}. Hence, D_3 is quasi-analytic.

Observe that H is invertible in $\{x+iy \in \mathbb{C}; x > 0\}$ and $H^{-1}(x+iy) = h^{-1}(x)+iy = (L \circ D_2 \circ E)(x) + iy$. Consider $\tilde{E}(x + iy) = \mathrm{Exp}(-x - iy)$ and define $F_2(x + iy) = \tilde{E}(H^{-1}(x + iy))$. Thus, F_2 is holomorphic in Δ_{b_1}, $F_2|_{\mathbb{R}} = f_2$. Notice that $F_2(x+iy) = D_2(E(x))(\cos y - \sin y)$ and $0 < D_2(E(x)) \leq 1$ for all $x > 0$. Therefore, F_2 is bounded in Δ_{b_1} and, consequently, D_2 is quasi-analytic. □

It concludes the proof of Theorem A.

4. Generalization and Proof of Theorem B

Consider $Z = (X, Y) \in \Omega^\omega$, remember that $p \in \Sigma = h^{-1}(0)$ is a fold point of X (resp. of Y) if $Xh(p) = 0$ and $X^2h(p) \neq 0$ (resp. $Yh(p) = 0$ and $Y^2h(p) \neq 0$). Moreover, a fold point of X (of Y), $p \in \Sigma$, is visible if $X^2h(p) > 0$ (resp. $Y^2h(p) < 0$) and it is invisible if $X^2h(p) < 0$ (resp. $Y^2h(p) > 0$).

Definition 5. *$p \in \Sigma$ is a fold-regular of $Z = (X, Y)$ if p is a fold of X (resp. of Y) and $Yh(p) \neq 0$ (resp. $Xh(p) \neq 0$). $p \in \Sigma$ is a fold-fold point of $Z = (X, Y)$ if p is, simultaneously, a fold of X and Y. $p \in \Sigma$ is a fold-saddle of Z if p is a fold of X (resp. of Y) and a saddle singularity of Y (resp. of X).*

As seen in the previous section, the proof of the quasi-regularity of a transition map does not have a drastic change if we consider hyperbolic polycycles of piecewise analytic vector fields. In this section we consider a weaker hypothesis by allowing fold-regular, fold-fold, and fold-saddle singularities in the polycycle, see Figure 3. The construction and analysis of the first return maps are done in the exactly same way as for the hyperbolic polycycles. Therefore, as we have seen above, it is enough to prove that the transition map, in a neighborhood a fold-regular or a fold-fold singularity, is quasi-analytic. The first step in this direction is to prove that transition maps at fold points are quasi-regular.

In order to study the quasi-regularity of the transition maps at fold points, we consider Σ^+ and Σ^- as being manifolds with boundary Σ which is locally, around the origin, given by $\Sigma = h^{-1}(0)$ where $h(x, y) = y$. Assume that X is a vector field defined in Σ^+ and Y is a vector field defined in Σ^-. In addition, assume the origin is a visible fold point of X and of Y, which are analytic vector fields. Then, from the approach developed in [20], there exists a neighborhood of the origin such that X and Y are separately conjugated (by means of $C^\infty-$diffeomorphisms) to

$$X_{ab}(x, y) = \begin{pmatrix} a \\ bx \end{pmatrix} \text{ and } Y_{cd}(x, y) = \begin{pmatrix} c \\ dx \end{pmatrix}, \tag{9}$$

respectively, with $ab > 0$ and $cd < 0$. Now, we define the transversal sections to study the transition maps: − For X_{ab}, consider $\varepsilon > 0$ and define

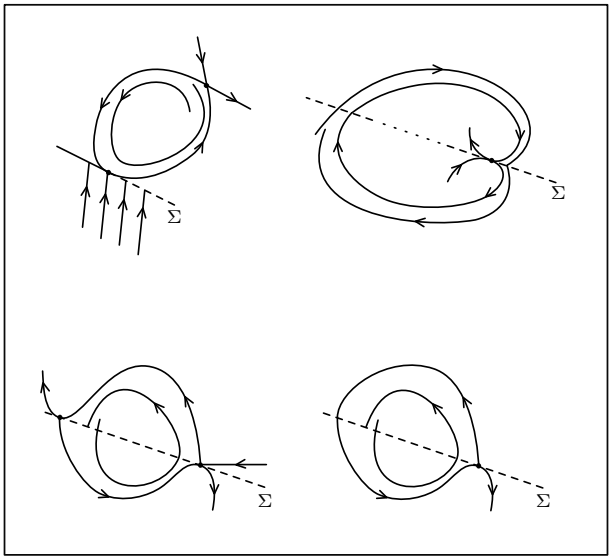

FIGURE 3. Examples of polycycles having singularities of the types saddle-fold, fold-regular and fold-folds.

- if $a > 0$: $\sigma \subset \Sigma$ and $\tau = \left\{(x,\varepsilon) \in \mathbb{R}^2; \sqrt{\frac{2a}{b}\varepsilon} - \delta < x < \sqrt{\frac{2a}{b}\varepsilon} + \delta\right\}$, for some $\delta > 0$. Moreover, if D^+_{ab} is the transition map at 0, in order to obtain $D^+_{ab}(0) = 0$, we consider another parametrization of τ, i.e., we consider the change $x \mapsto x - \sqrt{\frac{2a}{b}\varepsilon}$. Thus, by calculating the flow of X_{ab} we can also calculate the transition map from σ to τ, which is $D^+_{ab}(x) = \sqrt{x^2 + \frac{2a}{b}\varepsilon} - \sqrt{\frac{2a}{b}\varepsilon}$ for $x \geq 0$ with $(x,0) \in \sigma$;
- if $a < 0$: interchange the definition of σ and τ in the case $a > 0$. In this case, we need a new parametrization of σ and we obtain that by doing $x \mapsto x - \sqrt{\frac{2a}{b}\varepsilon}$ the transition map from σ to τ is $D^-_{ab}(x) = \sqrt{\left(x - \sqrt{\frac{2a}{b}\varepsilon}\right)^2 - \frac{2a}{b}\varepsilon}$ for $x \geq 0$ with $\left(x + \sqrt{\frac{2a}{b}\varepsilon}, \varepsilon\right) \in \sigma$.

– For Y_{cd}, consider $\varepsilon > 0$ and define

- if $c > 0$: $\sigma \subset \Sigma$ and $\tau = \left\{(x,-\varepsilon) \in \mathbb{R}^2; \sqrt{-\frac{2c}{d}\varepsilon} - \delta < x < \sqrt{-\frac{2c}{d}\varepsilon} + \delta\right\}$, for some $\delta > 0$. Analogously to the calculations for X_{ab} with $a > 0$ we obtain $D^+_{cd}(x) = \sqrt{x^2 + \frac{2c}{d}\varepsilon} - \sqrt{\frac{2c}{d}\varepsilon}$ for $x \geq 0$ with $(x,0) \in \sigma$;
- if $c < 0$: interchange the definition of σ and τ in the case $c > 0$. Analogously to the calculations for X_{ab} with $a < 0$ we obtain $D^-_{cd}(x) = \sqrt{\left(x - \sqrt{\frac{2c}{d}\varepsilon}\right)^2 - \frac{2c}{d}\varepsilon}$ for $x \geq 0$ with $\left(x + \sqrt{\frac{2c}{d}\varepsilon}, -\varepsilon\right) \in \sigma$.

See Figure 4 for a geometric illustration of the transition maps. Observe that, the case $a > 0$ (resp. $a < 0$) is analogous to the case $c > 0$ (resp. $c < 0$). Thus, it is enough to prove that $D^{\pm}_{ab}$ is quasi-regular.

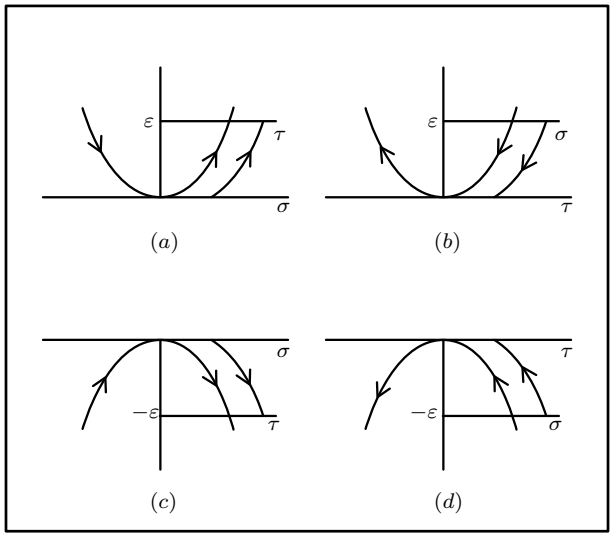

FIGURE 4. Transition maps through fold points: (a) X_{ab} with $a > 0$; (b) X_{ab} with $a < 0$; (c) X_{cd} with $c > 0$; (d) X_{cd} with $c < 0$.

Proposition 3. *Transition maps associated to visible fold points of C^∞ vector fields defined in a submanifold of $\mathbb{R}^2$ with boundary are quasi-regular homeomorphisms.*

Proof. Let the origin be a fold point of the C^∞ vector field X defined in $\Sigma^+ \cup \Sigma = \overline{\Sigma}$. As we have observed above, the case where the origin is a fold point for the vector field defined in Σ^- is analogous, then it is not considered.

Since X is $C^\infty-$conjugated to X_{ab} with $\Sigma = \{(x, 0) \in \mathbb{R}^2\}$, there exist $C^\infty-$ diffeomorphisms, φ and ψ, defined around the origin, with $\varphi(0) = \psi(0) = 0$ such that the transition map D, of X, satisfies

- $D(x) = \varphi \circ D^+_{ab} \circ \psi(x)$ if $a > 0$;
- $D(x) = \varphi \circ D^-_{ab} \circ \psi(x)$ if $a < 0$.

Since the set $\mathcal{D}$, of all quasi-regular homeomorphisms, is a group which contains the set Diff_0 of all diffeomorphisms fixing the origin, it is enough to show that $D^\pm_{ab}$ are quasi-regular homeomorphisms.

Observe that, for $a > 0$, D^+_{ab} can be $C^\infty-$extended to an open neighborhood of the origin. Then, we can calculate the infinity formal Taylor series

$$\sum_{i=1}^{\infty} \frac{1}{i!} \frac{d^i}{dx^i} D^+_{ab}(0) x^i,$$

where, for each $n \in \mathbb{N}$, we have $\left| D^+_{ab}(x) - \sum_{i=1}^{n} \frac{1}{i!} \frac{d}{dx} D^+_{ab}(0) x^i \right| = O(x^n)$. Moreover, $\frac{d}{dx} D^+_{ab}(0)$
$= 0$ and $\frac{d^2}{dx^2} D^+_{ab}(0) = \left(\frac{2a}{b}\varepsilon\right)^{-\frac{1}{2}} > 0$. Therefore, we obtain an asymptotic Dulac series by doing $\lambda_i = i + 1$ and $P_i(u) \equiv \frac{1}{(i+1)!} \frac{d^{i+1}}{dx^{i+1}} D^+_{ab}(0)$ for all $i \geq 2$, and so

$$\widehat{D}^+_{ab}(x) = \sum_{i=1}^{\infty} x^{\lambda_i} P_i(\ln x),$$

with $P_1 \equiv \left(\frac{2a}{b}\varepsilon\right)^{-\frac{1}{2}} > 0$. Hence, D_{ab}^{+} is a quasi-regular homeomorphism. Now, observe that $D_{ab}^{+} \circ D_{ab}^{-}(x) \equiv x$, so $D_{ab}^{+} \circ D_{ab}^{-} = \mathrm{Id}$. Since the set $\mathcal{D}$ is a group and D_{ab}^{+} and Id are quasi-regular, we get that D_{ab}^{-} is a quasi-regular homeomorphism. Therefore, we have proved the result. □

Now, we proceed to prove the quasi-analyticity of the transition regular map at fold-regular points.

Proposition 4. *Transition maps associated to visible fold points are quasi-analytic.*

Proof. It is enough to show that D_{ab}^{+} is quasi-analytic and from Proposition 3 it is enough to show the second part of Definition 4.

Let m be any positive real number and consider $g(x) = D_{ab}^{+}(e^{-x}) = \sqrt{e^{-2x} + \frac{2a}{b}\varepsilon} - \sqrt{\frac{2a}{b}\varepsilon}$. Since e^{-x} is a decreasing map, we have that $e^{-x} < e^{-m}$ for all $x > b$, moreover $e^{-x} \to 0$ as $x \to +\infty$. Therefore, e^{-2x} is bounded in the set $\{x \in \mathbb{R}; x > m\}$. Now, define the following complex extension of g,

$$\begin{array}{rccc} G: & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \mapsto & \sqrt{e^{-2z} + \frac{2a}{b}\varepsilon} - \sqrt{\frac{2a}{b}\varepsilon} \end{array}.$$

As $a, b, \varepsilon > 0$, G is a composition of holomorphic maps, G is thus a holomorphic map. Observe that, for $z = u + iv \in \mathbb{C}$, $|e^{-2z}| = |e^{-2u-2vi}| = |e^{-2u}(\cos(2v) - i\sin(2v))| \leq e^{-2u}$. Moreover, $\Delta_m = \{z = u + iv; u > m(1+v^2)^{1/4}\} \subset \{z = u + iv; u > m\}$. Therefore, G is bounded in the set $\Delta_m = \{z = u+iv; u > m(1+v^2)^{1/4}\}$ and it concludes the proof. □

Now, a straight consequence of this result and from the fact $\mathcal{D}$ is a group.

Corollary 1. *Transition maps associated to fold-regular, fold-fold or fold-saddle points of piecewise analytic vector fields, are quasi-regular homeomorphisms and quasi-analytic maps.*

From Proposition 4 and Corollary 1 we conclude the result obtained for hyperbolic polycycles does not change if fold points are allowed instead of saddle points. It concludes the proof of Theorem B.

References

1. A. A. Andronov, A. A. Vitt, and S. E. Khaikin, *Theory of oscillators*, Translated from the Russian by F. Immirzi; translation edited and abridged by W. Fishwick, Pergamon Press, Oxford-New York-Toronto, Ont., 1966.
2. R. Bamón, *Quadratic vector fields in the plane have a finite number of limit cycles*, Publications Mathématiques de l'IHÉS **64** (1986), 111–142.
3. E. A. Barbashin, *Introduction to the theory of stability*, Wolters-Noordhoff, 1970.
4. A. D. Bazykin, A. I. Khibnik, and B. Krauskopf, *Nonlinear dynamics of interacting populations*, vol. 11, World Scientific, 1998.
5. P. Bonckaert, *On the continuous dependence of the smooth change of coordinates in parametrized normal form theorems*, Journal of Differential Equations **106** (1993), no. 1, 107–120.
6. B. Brogliato, *Nonsmooth mechanics: models, dynamics and control*, Springer Science & Business Media, 2012.
7. K. da S. Andrade, *On degenerate cycles in discontinuous vector fields and the dulac's problem*, Ph.D. thesis, Unicamp, Brazil, 2016.
8. J. Ecalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, ActualitÃ©s MathÃ©matiques, Hermann, Paris, 1992.

9. A. F. Filippov, *Differential equations with discontinuous right-hand sides: control systems*, Mathematics and its Applications. Soviet Series, Kluwer Academic Publ, Dordrecht, 1988.
10. M. Guardia, T. M. Seara, and M. A. Teixeira, *Generic bifurcations of low codimension of planar Filippov systems*, J. Differential Equations **250** (2011), no. 4, 1967–2023.
11. C. Henry, *Differential equations with discontinuous right-hand side for planning procedures*, Journal of Economic Theory **4** (1972), no. 3, 545–551.
12. Y. S. Il'Yashenko, *Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane*, Functional Analysis and its Applications **18** (1984), no. 3, 199–209.
13. ———, *Finiteness theorems for limit cycles*, Translations of Mathematical Monographs, no. 94, American Mathematical Soc., 1991.
14. T. Ito, *A Filippov solution of a system of differential equations with discontinuous right-hand sides*, Economics Letters **4** (1979), no. 4, 349–354.
15. V. S. Kozlova, *Roughness of a discontinuous system*, Vestnik Moskovskogo Universiteta Seriya 1 Matematika Mekhanika (1984), no. 5, 16–20.
16. N. Minorsky and T. Teichmann, *Nonlinear oscillations*, Physics Today **15** (2009), no. 9, 63–65.
17. H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle (ii)*, Journal de Mathématiques Pures et Appliquées (1882), 251–296.
18. R. Roussarie, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields*, Bulletin of the Brazilian Mathematical Society **17** (1986), no. 2, 67–101.
19. ———, *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Springer, 1998.
20. S. M. Vishik, *Vector fields near the boundary of a manifold*, Vestnik Moskovskogo Universiteta Matematika **27** (1972), no. 1, 21–28.

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